The Dual Canonical Basis in the Spin Representation via the Temperley-Lieb Algebra

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Abstract

The spin representation $(\mathbb{C}^2)^{\otimes n}$ can be decomposed uniquely into irreducible modules called the Specht Modules, denoted W_k^n. Lusztig's dual canonical basis of the spin representation can be viewed as diagrams of the Temperley-Lieb algebra, and the Specht Modules also have a diagrammatic basis. We explicitly describe the decomposition of the spin representation into Specht Modules, $(\mathbb{C}^2)_k^{\otimes n} \cong W_n^n \oplus \cdots \oplus W_{n-2k}^n$, by computing the images of the diagrammatic basis elements of the Specht Modules. We do this by using induction to compute the image of W_n^n and reducing the problem of computing the images of other Specht modules to this case. Our results may lead to the notion of a canonical basis for Specht Modules in the future.

Summary

In both mathematics and physics, a common method of understanding complicated systems is to break them down into smaller pieces. Accordingly, it is known that a multiple-electron system can be viewed as multiple single-particle systems, but mathematicians and physicists do not yet know which single-particle systems a multiple-electron system explicitly breaks into. This paper investigates the question of computing such a decomposition explicitly. To do so, we investigate both the multiple-electron and single-particle systems mathematically and visually. We view multiple-electron systems using a mathematical object called the Temperley-Lieb algebra, an object defined in terms of diagrams with applications in statistical mechanics, representation theory, and topology. Similarly, we view the single-particle systems through certain diagrams of mathematical objects called Specht Modules. We compute how the Temperley-Lieb algebra explicitly relates to the Specht Modules through how the diagrams of both objects explicitly relate to each other. This, in turn, allows us to compute how a certain system with multiple electrons decomposes into systems with only one particle, simplifying computations in quantum mechanics.

1 Introduction

The spin representation $(\mathbb{C}^2)^{\otimes n}$ is an important object in various areas of mathematics and physics. Since it was first introduced by Cartan in 1913 [1] and applied to physics by Dirac in 1928 [2], mathematicians and physicists have studied it extensively. The spin representation parametrizes configurations of n electrons, each of which has spin-up or spin-down. We focus on $(\mathbb{C}^2)^{\otimes n}_k$, which parametrizes configurations with k spin-down electrons and $n-k$ spin-up electrons.

If one considers the spin representation $(\mathbb{C}^2)^{\otimes n}$ as a representation of a quantum group, the algebra of endomorphisms that commute with the action of a quantum group is precisely TL_n, the *Temperley-Lieb algebra* on n vertices. First introduced by Temperley and Lieb in 1971 [3], the Temperley-Lieb algebra has applications in statistical mechanics, representation theory, and topology. As a vector space, the Temperley-Lieb algebra is generated by diagrams consisting of two parallel lines with n vertices on each line, where each vertex is connected to exactly one other vertex and no lines intersect (see Figure 1).

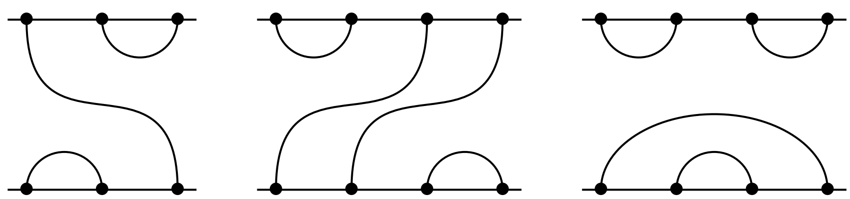

Figure 1: Examples of diagrams in the Temperley-Lieb algebra.

The spin representation with k spin-down electrons, $(\mathbb{C}^2)^{\otimes n}_k$, decomposes uniquely into irreducible modules named Specht modules, which are a generalization of the irreducible representations of S_n introduced by Specht in 1935 [4]. One method often used to understand algebraic structures, such as modules, is to find a diagrammatic interpretation that allows mathematicians to visualize the object combinatorially. Martin [5] discovered a diagrammatic basis for the Specht modules: the Specht module W^n_k for $k \equiv n \pmod 2$ can be expressed as a vector space spanned by certain diagrams from n vertices to k vertices.

Similarly, Khovanov [6] first introduced the possibility that the spin representation may have a diagrammatic interpretation. The author [7] generalized his work, allowing one to view the spin representation, specifically Lusztig's canonical basis [8], in terms of the diagrams of the Temperley-Lieb algebra.

This paper investigates the natural question of how to explicitly compute the decomposition of the spin representation into Specht modules in terms of the diagrams. Specifically, there exists a known isomorphism $\varphi_k^n\colon W_n^n \oplus W_{n-2}^n \oplus \cdots \oplus W_{n-2k}^n \to (\mathbb{C}^2)_k^{\otimes n}$, which we describe explicitly. We compute the images of each of the basis elements under φ_k^n in W_{n-2i}^n for $0 \leq i \leq k$, where $k = 0$ or $k = 1$, in Section 3, laying the groundwork for future induction. For $k = 1$, the solution to the more complicated case is that the image of the diagram $\mathrm{id}_n \in W_n^n$ is equal to $\sum_{i=0}^{n-1}[n-i]_q D_i$, where the diagrams D_i are shown in Figure 2.

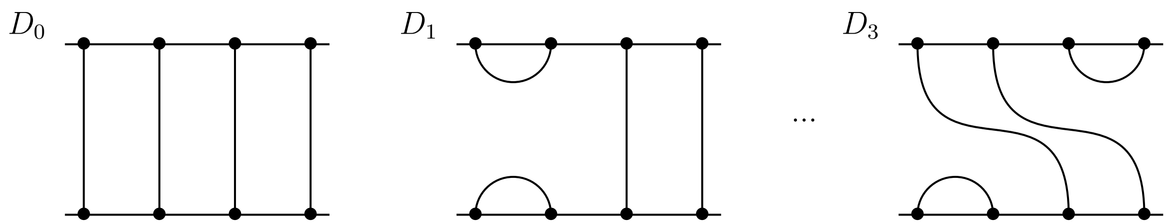

Figure 2: The diagrams D_i for $n = 4$, the diagrams in the image of id_n.

In Section 4, we describe a method to reduce the computation of images of each of the basis elements of W_{n-2i}^n for $1 \leq i \leq k$ to the computation of the image of a basis element in W_{n-2i}^{n-2i}. In Section 5, we explicitly describe the decomposition $(\mathbb{C}^2)_k^{\otimes n-1} \oplus (\mathbb{C}^2)_{k-1}^{\otimes n-1} \cong (\mathbb{C}^2)_k^{\otimes n}|_{\mathrm{TL}_{n-1}}$ and lay the groundwork for Section 6, in which we describe inductive formulas to explicitly compute the decomposition for general k. Specifically, the combination of Theorem 4.9 and Theorem 6.9 allows us to explicitly compute $\varphi_k^n\colon W_n^n \oplus W_{n-2}^n \oplus \cdots \oplus W_{n-2k}^n \to (\mathbb{C}^2)_k^{\otimes n}$ for all k and n.

As Chen [7] gives a diagrammatic description of Lusztig's dual canonical basis, a possible application of this research is to use this explicit decomposition to define the notion of a canonical basis for the Specht modules by having the Specht module inherit the canonical basis. Furthermore, the explicit decomposition of the spin representation allows us to view systems with many electrons as multiple one-particle systems, simplifying computations in quantum mechanics.

2 Background

2.1 The Temperley-Lieb Category

In this section, we introduce the Temperley-Lieb category. We also introduce notation that will be used throughout our paper.

Definition 2.1. A (m, n)-*diagram* comprises two parallel lines with m vertices on the bottom line and n vertices on the top line such that the vertices are connected by edges satisfying the following properties:

- the edges are between the parallel lines,
- the edges do not intersect each other, and
- each vertex is the endpoint of exactly one edge.

Given a (ℓ, m)-diagram and a (m, n)-diagram, we can obtain a (ℓ, n)-diagram by concatenation. In concatenation, a contractible loop may appear; we remove it and multiply the resulting diagram by formal variable $\beta = -q - q^{-1}$, where $q \in \mathbb{C}$ is not a root of unity. An example of concatenation is shown in Figure 3.

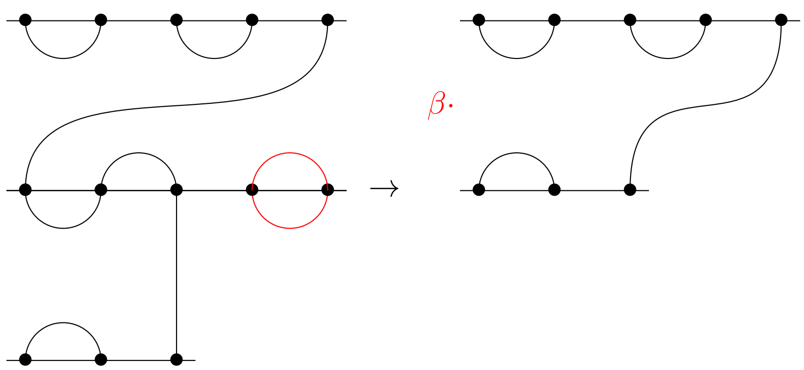

Figure 3: The composition of a $(3, 5)$-diagram and a $(5, 5)$-diagram.

Definition 2.2 (Temperley-Lieb Category)**.** The *Temperley-Lieb category*, denoted TL, has objects $\{[n], n \in \mathbb{Z}_{\geq 0}\}$, where $[n]$ is a set of n vertices. The morphisms $\mathrm{Hom}([m], [n])$ is the vector space with a basis of diagrams consisting of (m, n)-diagrams.

We call the edges of a diagram *links*. We say a link is

- *straight* if it connects the ith vertex on the bottom line to the ith vertex on the top line,
- *quasi-simple* if it connects two vertices on the same line, and
- *simple* if it connects two adjacent vertices on the same line.

Define $\epsilon_i^n \in \mathrm{Hom}([n],[n-2])$ to be the diagram connecting the ith and $(i+1)$th vertices on the bottom line and containing no other quasi-simple links. Similarly, define $\delta_i^n \in \mathrm{Hom}([n-2],[n])$ to be the diagram connecting the ith and $(i+1)$th vertices on the top line with no other quasi-simple links. An example of these diagrams are shown in Figure 4.

Figure 4: Diagrams for ϵ_2^4 and δ_2^4, respectively, in $\mathrm{Hom}([4],[2])$.

Every diagram in the Temperley-Lieb category can be decomposed into certain generators, as described by the following theorem.

Theorem 2.3 ([9])**.** *The algebra* $\bigoplus_{m,n\in\mathbb{Z}_{\geq 0}} \mathrm{Hom}([m],[n])$ *is generated as an algebra by all* ϵ_i^n *and* δ_i^n*, where* $1 \leq i \leq n-1$*, with the following relations:*

(i) $\epsilon_i^n \cdot \delta_i^n = -q - q^{-1}$*, and*

(ii) $(\mathrm{id} \otimes \epsilon_i^n) \cdot (\delta_i^n \otimes \mathrm{id}) = \mathrm{id} = (\epsilon_i^n \otimes \mathrm{id}) \cdot (\mathrm{id} \otimes \delta_i^n)$.

2.2 The Temperley-Lieb Algebra

In this section, we introduce the Temperley-Lieb Algebra, the object that is central to our paper. Refer to an (n,n)-diagram as an n-diagram.

Definition 2.4 (Temperley-Lieb Algebra)**.** Let the *Temperley-Lieb algebra*, denoted TL_n, be $\mathrm{Hom}([n],[n])$, the vector space spanned by diagrams with n vertices on both the top and bottom lines. Let tl_n denote the set of diagrams of TL_n.

Let e_i refer to the diagram with a simple link connecting the ith and $(i+1)$th vertices, counting from the left, on each line and with straight links everywhere else (for example, see Figure 5).

As a corollary of Theorem 2.3, we can state the following Proposition, describing the Temperley-Lieb algebra in terms of generators. Recall that $\beta = -q - q^{-1}$.

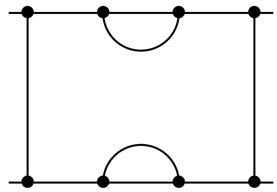

Figure 5: The diagram e_2 in TL_4.

Proposition 2.5 ([10])**.** *The algebra* TL_n *is generated by* $\{e_1, \dots, e_{n-1}\}$ *defined by the following relations:*

$$e_i^2 = \beta \cdot e_i,\ e_i e_{i\pm 1} e_i = e_i,\ \textit{and } e_i e_j = e_j e_i \textit{ for } |i-j| \geq 2.$$

2.3 The Spin Representation of the Temperley-Lieb Algebra

In this section, we discuss the spin representation $(\mathbb{C}^2)^{\otimes n}$, an important object in many areas of mathematics and physics, and how the Temperley-Lieb algebra acts on it.

Let v_+ denote $(1,0) \in \mathbb{C}^2$ and v_- denote $(0,1) \in \mathbb{C}^2$. The 4-dimensional vector space $\mathbb{C}^2 \otimes \mathbb{C}^2$ has basis $\{v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-\}$.

Let $\epsilon \colon \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}$ be

$$v_+ \otimes v_+ \mapsto 0, v_+ \otimes v_- \mapsto -q, v_- \otimes v_+ \mapsto 1, v_- \otimes v_- \mapsto 0$$

and let $\delta : \mathbb{C} \to \mathbb{C}^2 \otimes \mathbb{C}^2$ be

$$1 \mapsto v_+ \otimes v_- - q^{-1} v_- \otimes v_+.$$

Let $\epsilon_i^n \colon (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n-2}$ be $(\mathrm{id})^{\otimes i-1} \otimes \epsilon \otimes (\mathrm{id})^{\otimes n-i-1}$ and $\delta_i^n \colon (\mathbb{C}^2)^{\otimes n-2} \to (\mathbb{C}^2)^{\otimes n}$ be $\mathrm{id}^{\otimes i-1} \otimes \delta \otimes \mathrm{id}^{\otimes n-i-1}$.

Since ϵ_i^n and δ_i^n generate the algebra $\bigoplus_{m,n\geq 0} \mathrm{Hom}([m],[n])$ and all the relations in Proposition 2.5 are satisfied, we have an action of $\bigoplus_{m,n\geq 0} \mathrm{Hom}([m],[n])$ on $\bigoplus_{n\geq 0} (\mathbb{C}^2)^{\otimes n}$.

We can consider the case of $m = n$ to obtain a representation of TL_n.

Definition 2.6. The *spin representation* is the action of $\mathrm{TL}_n = \mathrm{Hom}([n],[n])$ on $(\mathbb{C}^2)^{\otimes n}$.

The spin representation is the central object of our paper.

Example 2.7. We describe an example of how TL_4 acts on $(\mathbb{C}^2)^{\otimes 4}$. Given the first diagram in Figure 6, we write it in terms of the generators of TL by decomposing it as shown in the second and third diagrams. Explicitly, the first diagram decomposes into $\delta_1^4 \delta_3^2 \epsilon_1^2 \epsilon_2^4$.

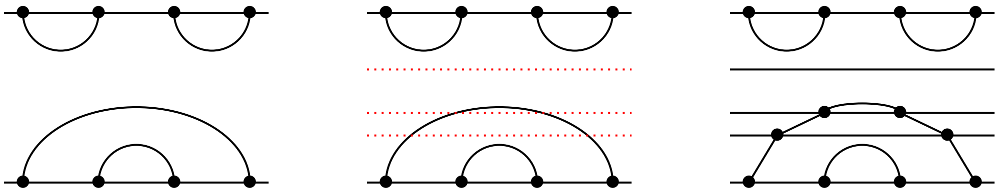

Figure 6: An example of how a TL_4 diagram is sliced.

We label each tensor product with its original position for convenience purposes. The action of the following diagram (Figure 6) is as follows:

$$\mathbb{C}^2_1 \otimes \mathbb{C}^2_2 \otimes \mathbb{C}^2_3 \otimes \mathbb{C}^2_4 \xrightarrow{\mathrm{id}\otimes\epsilon\otimes\mathrm{id}} \mathbb{C}^2_1 \otimes \mathbb{C}^2_4 \xrightarrow{\mathrm{id}\otimes\mathrm{id}} \mathbb{C}^2_2 \otimes \mathbb{C}^2_3 \xrightarrow{\epsilon} \mathbb{C} \xrightarrow{\delta\otimes\delta} \mathbb{C}^2_1 \otimes \mathbb{C}^2_2 \otimes \mathbb{C}^2_3 \otimes \mathbb{C}^2_4 \ .$$

2.4 A Diagrammatic Interpretation of the Spin Representation

Khovanov [6] introduced the possibility of a diagrammatic interpretation of the spin representation. Chen [7] generalized his work, representing Lusztig's dual canonical basis [8] as diagrams in the Temperley-Lieb algebra.

Definition 2.8. Let $(\mathbb{C}^2)^{\otimes n}_k$ denote the subspace of $(\mathbb{C}^2)^{\otimes n}$ generated by vectors of the form $v_{a_n} \otimes \cdots \otimes v_{a_1}$ such that k of the a_i are $-$ and $n-k$ of the a_i are $+$.

Remark. As the spin representation parametrizes the positions of n electrons, each of which has either spin-down or spin-up, the module $(\mathbb{C}^2)^{\otimes n}_k$ parametrizes the positions of n electrons where k of them have spin-down and $n-k$ have spin-up.

First, we define the notion of a diagrammatic basis for the induced representation that will be used to view the spin representation in terms of the diagrams of tl_n.

Lemma 2.9 ([7])**.** *For all $1 \leq k \leq n$, $\mathrm{Ind}^{\mathrm{TL}_n}_{\mathrm{TL}_k \otimes \mathrm{TL}_{n-k}} \mathbb{C}_{\mathrm{triv}}$ is precisely generated by the diagrams in which the only quasi-simple links on the bottom line connect $k-i$ to $k+1+i$ for some i, which is the diagrammatic basis of the left hand side.*

Definition 2.10. Let C^n_k denote the diagrammatic basis of $\mathrm{Ind}^{\mathrm{TL}_n}_{\mathrm{TL}_k \otimes \mathrm{TL}_{n-k}} \mathbb{C}_{\mathrm{triv}}$.

The following proposition describes how the spin representation can be viewed through diagrams of tl_n.

Proposition 2.11 ([7])**.** *There is an isomorphism of TL_n-modules:*

$$\mathrm{Ind}^{\mathrm{TL}_n}_{\mathrm{TL}_k \otimes \mathrm{TL}_{n-k}} \mathbb{C}_{\mathrm{triv}} \cong (\mathbb{C}^2)^{\otimes n}_k,$$

and the isomorphism identifies the diagrammatic basis of the left hand side to the dual canonical basis of the right hand side.

Therefore, $(\mathbb{C}^2)^{\otimes n}_k$ is generated by the diagrams in C^n_k.

We can extend Proposition 2.11 to the entire spin representation to obtain the following, giving a diagrammatic basis of the entire Temperley-Lieb algebra.

Corollary 2.12 ([7])**.** *There is an isomorphism of* TL_n*-modules*

$$\bigoplus_{0\le k\le n} \mathrm{Ind}^{\mathrm{TL}_n}_{\mathrm{TL}_k \otimes \mathrm{TL}_{n-k}} \mathbb{C}_{\mathrm{triv}} \cong (\mathbb{C}^2)^{\otimes n}$$

which identifies the diagrammatic basis of the left hand side to the dual canonical basis of the right hand side.

With these results, we can work with $(\mathbb{C}^2)^{\otimes n}$ in terms of the diagrams in tl_n.

2.5 Specht Modules

We now study the irreducible representations of $(\mathbb{C}^2)^{\otimes n}_k$, called the *Specht modules*, which are another important object in our work.

Definition 2.13. A diagram is *monic* when it does not have any simple links on the bottom line. For $0 \le \ell \le n$ such that $\ell \equiv n \pmod 2$ let w^n_ℓ denote the set of monic diagrams with n vertices on the top line and ℓ vertices on the bottom line.

Definition 2.14 (Specht modules)**.** The *Specht module* W^n_ℓ is the TL_n-module which is the quotient of $\mathrm{Hom}([\ell],[n])$ by the image of left multiplication by $\epsilon^{\ell-2}_j$s.

The following lemma gives an explicit description of the action of TL_n on the Specht Module.

Lemma 2.15. *The Specht module* W^n_ℓ *has a basis consisting of the diagrams in* w^n_ℓ*. A diagram* $D \in \mathrm{tl}_n$ *sends* $D' \in w^n_\ell$ *to the concatenation* DD' *if the* (n,ℓ)*-diagram* DD' *remains monic, and to* 0 *otherwise.*

Example 2.16. The composition of the two diagrams in Figure 7 is 0, as the monicity of the diagram in w^6_2 is not preserved.

The following Proposition can be obtained through results in [11].

Figure 7: A diagram in the Specht Module W_2^6 (left) and a diagram in the Temperley-Lieb algebra TL_6 (right).

Proposition 2.17 ([11])**.** *The* TL_n*-representation* $(\mathbb{C}^2)^{\otimes n}_k$ *for* $k \leq n/2$ *can be uniquely (up to scalar multiplication) decomposed into a direct sum of Specht modules* $W^n_n \oplus W^n_{n-2} \oplus \cdots \oplus W^n_{n-2k}$.

Proof. There is a decomposition of $(\mathbb{C}^2)^{\otimes n}$ as a $\mathrm{TL}_n \otimes \mathcal{U}_q(\mathfrak{sl}_2)$-module as[1]

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_{0 \leq k \leq n/2} W^n_{n-2k} \boxtimes (\mathbb{C}^2)^{\otimes n}_{n/2-k}. \tag{1}$$

Since $Kv_\pm = q^{\pm 1} v_\pm$, the q^ℓ-eigenspace of K in (1) gives that

$$(\mathbb{C}^2)^{\otimes n}_{n-2\ell} = \bigoplus_{0 \leq k \leq |\ell|} W^n_{n-2k}.$$

□

As both $(\mathbb{C}^2)^{\otimes n}_k$ and W^n_i for all $0 \leq i \leq k$ where $k \equiv n \pmod 2$ have a diagrammatic interpretation, a natural question to ask is how the above isomorphism interacts with the diagrammatic perspective. We investigate the images of the diagrams in w^n_i, leading to an explicit decomposition of the spin representation.

3 Computation of φ^n_k for $k = 0$ and $k = 1$

In the previous section, we have defined an isomorphism $\varphi^n_k\colon W^n_n \oplus W^n_{n-2} \oplus \cdots \oplus W^n_{n-2k} \to (\mathbb{C}^2)^{\otimes n}_k$. In this section, we describe φ^n_k explicitly in terms of diagrams for $k = 0$ and $k = 1$, laying the groundwork for an inductive computation for general k in Section 6. Recall that w^n_k denotes the diagrammatic basis of W^n_k, tl_n denotes the diagrammatic basis of TL_n, and C^k_n denotes the diagrammatic basis of $(\mathbb{C}^2)^{\otimes n}_k$. This notation will be used throughout the paper.

[1] where $\boxtimes$ denotes the box tensor product.

Lemma 3.1. *The image of the unique diagram in w_n^n under φ_0^n is equal to the identity diagram in* tl_n*.*

Proof. In both W_n and TL_n, the action of the generators e_i (see Figure 5) for all i send the diagram described in the lemma to 0. □

Recall that ϵ_i^n and δ_i^n (see Figure 4) are the generators of the Temperley-Lieb category.

Proposition 3.2. *We can describe the image of φ_1^n as follows:*

(i) the image of a diagram $D \in w_{n-2}^n$ is equal to $D\epsilon_1^{n-2}$; and

(ii) the image of the diagram $\mathrm{id}_n \in w_n^n$ is equal to $\sum_{i=0}^{n-1}[n-i]_q D_i$, where $D_i = \delta_i^n \cdot \epsilon_1^n$ for $1 \le i \le n-1$ and D_0 is the identity diagram in tl_n*.*

Proof. First, statement (i) is true as the actions of e_i for all i will be consistent across both W_{n-2}^n and $(\mathbb{C}^2)_1^{\otimes n}$.

Now, we prove statement (ii). The action of e_i on D_0 will give D_1 for $i = 1$ and 0 for $i \neq 1$. For each D_i for $1 \le i \le n-1$, we have that

$$e_j D_i = \begin{cases} \beta D_i & \text{if } j = i \\ D_i & \text{if } j = i \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let c_i denote the coefficient of D_i in $\varphi_1^n(\mathrm{id}_n)$. It is necessary that $e_j \varphi_1^n(\mathrm{id}_n) = e_j \sum_i c_i D_i = 0$ for all j. From the action of the e_i on the diagrams described above, we obtain that $c_{n-1} = 1$, $c_{n-2} = -\beta = q + q^{-1}$, and $c_{n-i} = (q+q^{-1})c_{n-i+1} - c_{n-i+2}$. Induction allows us to conclude that that $c_{n-i} = \frac{q^i - q^{-i}}{q - q^{-1}} = [i]_q$. Therefore, we obtain that $\varphi_1^n(\mathrm{id}_n) = \sum_{i=0}^{n-1}[n-i]_q D_i$. □

This proposition will be used in Section 6 as a base case for our induction procedure.

4 Reduction Procedure For General k

In this section, we describe how to calculate the images under φ_k^n of the diagrammatic basis of $W_{n-2}^n, \ldots, W_{n-2k}^n$ by reducing the problem to computing the images of the diagrammatic basis of $W_{n-2}^{n-2}, \ldots, W_{n-2k}^{n-2}$ under φ_{k-1}^{n-2}. This procedure can be applied repeatedly, allowing us to reduce the problem of computing $\varphi_k^n(W_{n-2i}^n)$ to computing $\varphi_{k-i}^{n-2i}(W_{n-2i}^{n-2i})$. We construct this reduction by proving this reduction for the specific case of diagrams with links in the top left corner in Proposition 4.7, and then extending it to all diagrams in Theorem 4.9.

The commutative diagram in Figure 8 summarizes the reduction procedure we describe in this section.

$$\begin{array}{ccc} W^{n-2i}_{n-2i} & \xrightarrow{\varphi^{n-2i}_{k-i}} & (\mathbb{C}^2)^{\otimes n-2i}_{k-i} \\ {\scriptstyle D\mapsto \Pi_j\delta_j\cdot D}\big\downarrow & & \big\downarrow{\scriptstyle D\mapsto \Pi_j\delta_k\cdot D\cdot \Pi^i_{j=0}\epsilon^{n-2i+2j}_k} \\ W^n_{n-2i} & \xrightarrow{\varphi^n_k} & (\mathbb{C}^2)^{\otimes n}_k \end{array}$$

Figure 8: How the computation of $\varphi^n_k(W^n_{n-2i})$ reduces to the computation of $\varphi^{n-2i}_{k-i}(W^{n-2i}_{n-2i})$.

Recall that $(\mathbb{C}^2)^{\otimes n-2}_{k-1} \cong W^{n-2}_{n-2} \oplus \cdots \oplus W^{n-2}_{n-2k}$ and $(\mathbb{C}^2)^{\otimes n}_k \cong W^n_n \oplus W^n_{n-2} \oplus \cdots \oplus W^n_{n-2k}$. The action of $D \in \mathrm{tl}_{n-2}$ on $\mathbb{C}^{\otimes n}_k$ is defined using the embedding $\mathrm{TL}_{n-2} \hookrightarrow \mathrm{TL}_n$ which sends e_i to e_{i+2}. An example of the embedding is in Figure 9.

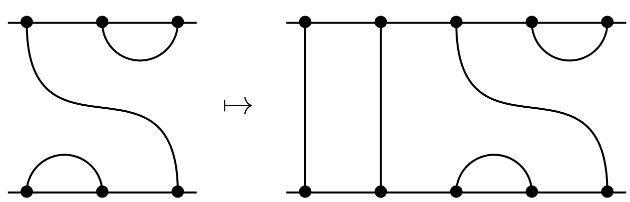

Figure 9: A basis element of TL_3 embedded into TL_5.

Definition 4.1. Define the map $\Phi\colon (\mathbb{C}^2)^{\otimes n-2}_{k-1} \to ((\mathbb{C}^2)^{\otimes n}_k)^{e_1=\beta}$ by $\Phi\colon D \mapsto \delta^n_1 \cdot D \cdot \epsilon^n_k$ for diagrams D in C^{n-2}_{k-1}.

The map Φ is a specific case of the rightmost map in Figure 8.

Lemma 4.2. *The map Φ is an isomorphism of TL_{n-2}-representations.*

Proof. We can verify that Φ is well-defined and commutes with the action of TL_{n-2}, so it is a homomorphism.

Let S be the set of diagrams D in C^n_k that can be expressed in the form $D = \delta^n_1 \cdot D'$, where D' is any diagram in $\mathrm{Hom}([n],[n-2])$ in TL (in other words, D contains a link in the top left corner). Let R be the set of diagrams in the same set C^n_k that cannot be expressed in this way (in other words, the diagrams in R do not have a link in the top left corner).

A vector v in $((\mathbb{C}^2)^{\otimes n}_k)^{e_1=\beta}$ can be expressed as $\sum_{D\in S} a_D D + \sum_{A\in R} b_A A$, where the a_D and b_A are coefficients in $\mathbb{C}^2$. As every individual component $a_D D$ has eigenvector e_1 with eigenvalue β, $\sum_{A\in R} b_A A$ must also be e_1-invariant with eigenvalue β. However, e_1 sends any diagram $A \in R$ to a diagram in S, which implies that $\sum_{A\in R} b_A A = 0$, so v is spanned by diagrams in S.

We now show that Φ is surjective. Every diagram in S has e_1 as a eigenvector with eigenvalue β, so we conclude that $((\mathbb{C}^2)_k^{\otimes n})^{e_1=\beta}$ is generated by diagrams of the form $\delta_1^n \cdot D$. Furthermore, as such a diagram in $\mathrm{Ind}_{\mathrm{TL}_k \otimes \mathrm{TL}_{n-k}}^{\mathrm{TL}_n} \mathbb{C}_{\mathrm{triv}}$ must also have a link on the bottom line connecting the kth and $(k+1)$th vertices, it can also be expressed in the form $\delta_1^n \cdot D \cdot \epsilon_k^n$.

Injectivity follows from construction, so Φ is an isomorphism. □

Remark. The map $\Phi\colon D \mapsto \delta_1^n \cdot D \cdot \epsilon_k^n$ in Lemma 4.2 is adding a link to the top left and a link on the bottom line between the kth and $(k+1)$th vertices of D.

Example 4.3. The diagram on the left maps under Φ to the diagram in the right in Figure 10.

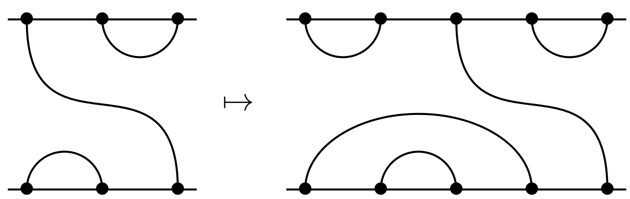

Figure 10: An example of a map from a diagram in $(\mathbb{C}^2)_2^{\otimes 3}$ to $(\mathbb{C}^2)_2^{\otimes 5}$ under Φ.

Definition 4.4. For each $1 \le i \le n/2$, define $\Phi_{n-2i}\colon W_{n-2i}^{n-2} \to (W_{n-2i}^n)^{e_1=\beta}$ by

$$\Phi_{n-2i}\colon D \mapsto \delta_1^n \cdot D.$$

This is a specific case of the leftmost map in Figure 8.

Lemma 4.5. *The map Φ_{n-2i} is an isomorphism of TL_n-modules for all i.*

Proof. Using a method similar to the proof of Lemma 4.2, we can verify that Φ_{n-2i} is both well-defined and intertwining with the TL_{n-2} action for all i, so it is a homomorphism.

Let S be the set of diagrams D in w_{n-2i}^n that can be expressed as $\delta_1^n \cdot D$ where $D \in w_{n-2i}^{n-2}$ (so S is the set of diagrams that have a link in the top left corner). Let R be the set of diagrams in w_{n-2i}^n that cannot be expressed this way. A vector v in $(W_{n-2i}^n)^{e_1=\beta}$ can be expressed as $\sum_{D\in S} a_D D + \sum_{A\in R} b_A A$. As every individual component $a_D D$ has e_1 as an eigenvector with eigenvalue β, $\sum_{A\in R} b_A A$ must also have β as the e_1-eigenvalue. The action of e_1 on any diagram in w^n_{n-2i} must map to either zero or a diagram in S, so it cannot send $A \in R$ to βA. This implies that $\sum_{A\in R} b_A A = 0$, so v is spanned by diagrams in S. Furthermore, every diagram in S has a e_1-eigenvalue β, so we conclude that $(W^n_{n-2i})^{e_1=\beta}$ is generated by diagrams that can be expressed in the form $\delta^n_1 \cdot D$.

The above implies that Φ_{n-2i} is surjective, and injectivity follows from construction, so Φ_{n-2i} is an isomorphism. □

Remark. The map $\Phi_{n-2i}\colon D \mapsto \delta^n_1 \cdot D$ in Lemma 4.5 is adding a link to the top left corner of D.

In addition to Lemma 4.2 and Lemma 4.5, we can use the following general result, which follows from Schur's Lemma.

Lemma 4.6 (Schur's Lemma)**.** *Given irreducible representations $A_1, \dots, A_r$ and $A'_1, \dots, A'_r$ of TL_n such that $A_i \ncong A_j$ and $A'_i \ncong A'_j$ when $i \neq j$ and $\varphi_i\colon A_i \cong A'_i$ is an isomorphism for all i, if $\varphi\colon A_1 \oplus \cdots \oplus A_r \cong A'_1 \oplus \cdots \oplus A'_r$ is an isomorphism, then $\varphi|_{A_i}$ lands in A'_i and is a multiple of φ_i.*

We can now prove the following proposition, which reduces the problem of computing the images of diagrams with links on the top left corner in w^n_{n-2i} to computing the images of diagrams in w^{n-2}_{n-2i}. This proposition is generalized in Theorem 4.9.

Proposition 4.7. *The image of the isomorphism $\Phi\colon (\mathbb{C}^2)^{\otimes n-2}_{k-1} \to ((\mathbb{C}^2)^{\otimes n}_k)^{e_1=\beta}$ of each component W^{n-2}_{n-2i} of $(\mathbb{C}^2)^{\otimes n-2}_{k-1}$ falls in W^n_{n-2i}. Furthermore, the restriction of Φ for each component W^{n-2}_{n-2i} is equal to the map Φ_{n-2i}.*

Proof. Lemma 4.2 gives an isomorphism $\Phi\colon (\mathbb{C}^2)^{\otimes n-2}_{k-1} \to ((\mathbb{C}^2)^{\otimes n}_k)^{e_1=\beta}$ and Lemma 4.5 give isomorphisms between irreducible representations $\Phi_{n-2}\colon W^{n-2}_{n-2} \to (W^n_{n-2})^{e_1=\beta}$, $\dots$, $\Phi_{n-2k}\colon W^{n-2}_{n-2k} \to (W^n_{n-2k})^{e_1=\beta}$. Lemma 4.6 implies that the restriction $\Phi|_{W^n_{n-2i}}$ is precisely Φ_{n-2i}. □

From this inductive procedure, one can also compute the images of every diagram in $w^n_{n-2}, \dots, w^n_{n-2k}$ under φ^n_k given the images of the diagrams in $w^{n-2}_{n-2}, \dots, w^{n-2}_{n-2k}$ under φ^{n-1}_{k-1}.

Lemma 4.8. *Given any diagram $\delta^n_i \cdot D$ in either a Specht module or the Temperley-Lieb algebra, the action of e_{i+1} sends it to $\delta^n_{i+1} \cdot D$.*

Proof. This lemma follows by the fact that $e_{i+1}\delta_i = \delta_{i+1}$. □

The following theorem is the main result of this section, allowing us to reduce the problem of computing $\varphi_k^n(W_{n-2i}^n)$ to computing $\varphi_{k-i}^{n-2i}(W_{n-2i}^{n-2i})$.

Theorem 4.9. *Let $D = \delta_j^n D'$, where $D' \in w_{n-2i}^{n-2}$, be a diagram in w_{n-2i}^n for some $1 \leq i \leq k$. Let $\varphi_{k-1}^{n-2}(D') = \sum_{A \in \mathrm{tl}_{n-2}} c_A A$. Then*

$$\varphi_k^{n-2}(D) = \sum_{A \in \mathrm{tl}_{n-2}} c_A \delta_j^n \cdot A \cdot \epsilon_k^n.$$

Proof. Lemma 4.8 implies that given $\varphi_k^n(\delta_i^j \cdot D)$ for any $\delta_i^j \cdot D \in (W_{n-2i}^n)^{e_j=\beta}$, one can obtain an element in $(W_{n-2i}^n)^{e_{j+1}=\beta}$ by $\delta_j^n D \mapsto \delta_{j+1}^n D$. Note that this operation gives an isomorphism of vector spaces $(W_{n-2i}^n)^{e_j=\beta} \cong (W_{n-2i}^n)^{e_{j+1}=\beta}$, so every element of $(W_{n-2i}^n)^{e_{j+1}=\beta}$ can be obtained this way.

As Proposition 4.7 gives the image of $(W_{n-2i}^n)^{e_1=\beta}$ under φ_k^n, we can compute the image of $(W_{n-2i}^n)^{e_j=\beta}$ under φ_k^n for all $1 \leq j \leq n-1$, yielding the explicit irreducible decomposition of $(\mathbb{C}^2)_k^{\otimes n}$. □

With this theorem, we obtain that for $i = 1$, Figure 8 commutes. We can repeatedly apply this theorem to obtain the reduction described in the diagram for general i.

Remark. In this section, we reduced the problem of computing the images of the diagrams in w_{n-2i}^n with links in the top left corner, and we reduced the problem of computing the images of the other diagrams of w_{n-2i}^n through applying actions of e_j to the former case. Alternatively, we could have also directly computed these using reduction, as $(\mathbb{C}^2)_{k-1}^{n-2}$ is isomorphic to $((\mathbb{C}^2)_k^n)^{e_j=\beta}$ as $\mathrm{TL}_{i-1} \otimes \mathrm{TL}_{n-i-1}$-modules.

5 The Decomposition $(\mathbb{C}^2)_k^{\otimes n-1} \oplus (\mathbb{C}^2)_{k-1}^{\otimes n-1} \cong (\mathbb{C}^2)_k^{\otimes n}|_{\mathrm{TL}_{n-1}}$

The spin representation can be decomposed into a direct sum of "smaller" spin representations. In this section, we describe the isomorphism $(\mathbb{C}^2)_k^{\otimes n-1} \oplus (\mathbb{C}^2)_{k-1}^{\otimes n-1} \cong (\mathbb{C}^2)_k^{\otimes n}|_{\mathrm{TL}_{n-1}}$ explicitly, using it to give an inductive procedure to compute $\varphi_k^n(W_n^n)$ in the next section.

We use the following result from [12] in order to prove Proposition 5.2, which allows us to decompose the spin representation into spin representations of a smaller number of electrons, thus allowing us to describe an inductive procedure to calculate the image of the generator of $\varphi_k^n(W_n^n)$ in Section 6.

Lemma 5.1 ([12])**.** *There is an isomorphism of TL_{n-1}-modules $W_k^n|_{\mathrm{TL}_{n-1}} \cong W_{k-1}^{n-1} \oplus W_{k+1}^{n-1}$.*

Proposition 5.2. *There is an isomorphism of* TL_{n-1}*-modules* $(\mathbb{C}^2)^{\otimes n}_k|_{\mathrm{TL}_{n-1}} \cong (\mathbb{C}^2)^{\otimes n-1}_k \oplus (\mathbb{C}^2)^{\otimes n-1}_{k-1}$.

Proof. Proposition 2.17 states that $(\mathbb{C}^2)^{\otimes n}_k \cong W^n_n \oplus \cdots \oplus W^n_{n-2k}$. Restricting the action on both sides to TL_{n-1}, we obtain from Lemma 5.1 that $(\mathbb{C}^2)^{\otimes n}_k$ is isomorphic to

$$\begin{aligned} W^{n-1}_{n-1} \oplus W^{n-1}_{n+1} \cdots \oplus W^{n-1}_{n-2k-1} \oplus W^{n-1}_{n-2k+1} &= W^{n-1}_{n-1} \oplus W^{n-1}_{n-4} \oplus W^{n-1}_{n-2} \cdots \oplus W^{n-1}_{n-2k-1} \oplus W^{n-1}_{n-2k+1} \\ &= (\mathbb{C}^2)^{\otimes n-1}_k \oplus (\mathbb{C}^2)^{\otimes n-1}_{k-1}. \end{aligned}$$

□

We describe one such isomorphism explicitly.

Definition 5.3. We define the map $\Theta^n_k \oplus \Psi^n_k : (\mathbb{C}^2)^{\otimes n-1}_k \oplus (\mathbb{C}^2)^{\otimes n-1}_{k-1} \to (\mathbb{C}^2)^{\otimes n}_k|_{\mathrm{TL}_{n-1}}$ on each component.

- Let Θ^n_k be defined by mapping a diagram $D \in C^{n-1}_k$ to the diagram D with a link added to the right.
- Define Ψ^n_k as mapping the identity diagram id_{n-1} to $\delta^n_k\epsilon^n_k + [2]_q\delta^n_{k+1}\epsilon^n_k + \cdots + [n-k]_q\delta^n_{n-1}\epsilon^n_k$. Extend it to be compatible with the TL_{n-1}-action.

In this section, we show that $\Theta^n_k \oplus \Psi^n_k$ is an isomorphism. First, we show that it is a homomorphism.

Proposition 5.4. *The maps* Θ^n_k *and* Ψ^n_k *are well-defined homomorphisms.*

Proof. Firstly, Θ^n_k is a homomorphism of TL_{n-1}-modules. Indeed, adding a link at the end does not affect any actions or relations.

We prove that the actions of $e_2, \ldots, e_{n-1}$ send $\Phi^n_k(\mathrm{id}_{n-1})$ to 0. First, we consider e_k. The action annihilates all of $\delta^n_{k+2}\epsilon^n_k$ to $\delta^n_{n-1}\epsilon^n_k$ as it creates a link on the bottom line between the $(k+2)$th and $(k+3)$th vertices. What remains is $e_k(\delta^n_k\epsilon^n_k + [2]_q\delta^n_{k+1}\epsilon^n_k) = \beta e_k + [2]_q e_k = 0$. Now we consider e_i for $i > k$. The action annihilates everything except for $[i-k]_q\delta^n_{i-1}\epsilon^n_k + [i-k+1]_q\delta^n_i\epsilon^n_k$. Furthermore, $e_i([i-k]_q\delta^n_{i-1}\epsilon^n_k + [i-k+1]_q\delta^n_i\epsilon^n_k + [i-k+2]_q\delta^n_{i+1}\epsilon^n_k) = [i-k]_q\delta^n_i\epsilon^n_k + [i-k+1]_q\beta\delta^n_i\epsilon^n_k + [i-k+2]_q\delta^n_i\epsilon^n_k = 0$.

As $(\mathbb{C}^2)^{\otimes n-1}_k \cong \mathrm{Ind}^{\mathrm{TL}_n}_{\mathrm{TL}_k \otimes \mathrm{TL}_{n-k}} \mathbb{C}_{\mathrm{triv}}$, the universal property of induced representations states that there are no further relations. □

We compute the image of certain diagrams under Ψ^n_k, which will be used in Section 6.

Lemma 5.5. *The map* Ψ^n_k *sends* $\delta^{n-1}_i\epsilon^{n-1}_{k-1}$ *to* $\delta^n_i\epsilon^n_{k-1}(\delta^n_k + [2]_q\delta^n_{k+1} + \cdots + [n-k]_q\delta^n_{n-1})\epsilon^n_k$.

Proof. To obtain the image of $\delta_{k-1}^{n-1}\epsilon_{k-1}^{n-1}$ under Ψ_k^n, we apply e_{k-1} to id_{n-1} and compute its image. The image will be $\delta_{k-1}^n\epsilon_{k-1}^n(\delta_k^n\epsilon_k^n+[2]_q\delta_{k+1}^n\epsilon_k^n+\cdots+[n-k]_q\delta_{n-1}^n\epsilon_k^n)$ by Definition 5.3. □

Finally, we prove the following result, which allows us to use induction to compute $\varphi_k^n(W_n^n)$ in Section 6.

Theorem 5.6. *The map $\Theta_k^n \oplus \Psi_k^n$ defines an isomorphism of TL_{n-1}-modules.*

Proof. Each of the diagrams in C_k^{n-1} maps to a unique diagram in C_k^n with a straight link on the nth vertex. The last term the images of each of the diagrams in C_{k-1}^{n-1} gives a diagram $\delta_i^n\epsilon_{k-1}^n\delta_{n-1}^n\epsilon_k^n$ that does not appear in the image of any other diagram under $\Theta_k^n\oplus\Psi_k^n$. Therefore, each of the images are linearly independent. □

6 Computation of $\varphi_k^n(W_n^n)$ Using the Decomposition

Recall that $\Theta_k^n \oplus \Psi_k^n : (\mathbb{C}^2)_k^{\otimes n-1} \oplus (\mathbb{C}^2)_{k-1}^{\otimes n-1} \to (\mathbb{C}^2)_k^{\otimes n}|_{\mathrm{TL}_{n-1}}$ is an isomorphism by Theorem 5.6.

We state the following proposition, which we use to inductively compute $\varphi_k^n(W_n^n)$.

Proposition 6.1. *Given vectors $x_k^{n-1} \in \varphi_k^{n-1}(W_{n-1}^{n-1})$ in $(\mathbb{C}^2)_k^{\otimes n-1}$ and $x_{k-1}^{n-1} \in \varphi_{k-1}^{n-1}(W_{n-1}^{n-1})$ in $(\mathbb{C}^2)_{k-1}^{\otimes n-1}$ that are annihilated by the actions of $e_1, \ldots, e_{n-2}$, the one-dimensional space $\varphi_k^n(W_n^n)$ is generated by the unique (up to scalar) vector in $(\mathbb{C}^2)_k^{\otimes n}$ that is annihilated by the actions of all e_is. This vector is the unique (up to scalar) linear combination of $\Theta_k^n(x_k^{n-1})$ and $\Psi_k^n(x_{k-1}^{n-1})$ that is also annihilated by e_{n-1}.*

Proof. The image of the identity diagram id_n in W_n^n under the homomorphism φ, is the unique vector (up to scalar) that is annihilated by the actions of all e_is. This vector is in the space V of vectors that are annihilated by the actions of e_i for $1 \le i \le n-2$. Due to the TL_{n-1}-module isomorphism $\Theta_k^n \oplus \Psi_k^n$, the subspace V is 2-dimensional, generated by $\Theta_k^n(x_k^{n-1})$ and $\Psi_k^n(x_{k-1}^{n-1})$. Therefore, the vector in $(\mathbb{C}^2)_k^{\otimes n}$ that is annihilated by the actions of all e_is is a linear combination of $\Theta_k^n(x_k^{n-1})$ and $\Psi_k^n(x_{k-1}^{n-1})$. □

We construct a sequence x_i^n, in which each term is a linear combination of the images of the two before, and each term is annihilated by the action of every generator.

Definition 6.2. Let $x_0^n = x_n^n = \mathrm{id}_n$ and define x_k^n inductively by $[n-k]_q x_k^n = [n]_q\Theta_k^n(x_k^{n-1}) + \Psi_k^n(x_{k-1}^{n-1})$.

We show that x_k^n is the unique vector that is annihilated by all of $e_1, \ldots, e_{n-1}$ in $(\mathbb{C}^2)_k^{\otimes n}$. Note that for $k = 0$, id_n is annihilated by the action of every generator, so this statement holds. In this section, we inductively prove that x_k^n defined this way is annihilated by the action of all the generators, for general k.

6.1 The $k = 1$ Case

We first show that x_1^n is annihilated by the action of all e_is. To do so, we introduce the following lemma.

Lemma 6.3. *We have the relation* $[n]_q[n-i-1]_q + [i]_q = [n-1]_q[n-i]_q$.

Proof. We have the above is equal to

$$\begin{aligned}
[n]_q[n-i-1]_q + [i]_q &= \frac{q^n - q^{-n}}{q - q^{-1}} \cdot \frac{q^{n-i-1} - q^{i+1-n}}{q - q^{-1}} + \frac{q^i - q^{-i}}{q - q^{-1}} \\
&= \frac{q^{2n-i-1} - q^{i+1} - q^{-i-1} + q^{i+1-2n} + q^{i+1} - q^{-i+1} - q^{-1+i} + q^{-i-1}}{(q - q^{-1})^2} \\
&= \frac{q^{2n-i-1} + q^{i+1-2n} - q^{-i+1} - q^{-1+i}}{(q - q^{-1})^2} \\
&= \frac{(q^{n-1} - q^{1-n})(q^{n-i} - q^{i-n})}{(q - q^{-1})^2} \\
&= [n-1]_q[n-i]_q.
\end{aligned}$$

□

With this, we can show that the definition of x_1^n is consistent for the $k = 1$ case.

Proposition 6.4. *The inductive computation of* x_1^n *by* $[n-1]_q x_1^n = [n]_q \Theta_1^n(x_1^{n-1}) + \Psi_1^n(x_0^{n-1})$ *yields that* x_1^n *is annihilated by the action of all the generators.*

Proof. We give a formula for x_1^n by induction. In the base case, we have that $\Theta_1^1(x_1^1) = \mathrm{id}_1$.

By the inductive hypothesis, we have

$$[n]_q \Theta_1^n(x_1^{n-1}) = [n]_q \left(\sum_{i=1}^{n-2} [n-i-1]_q \delta_i^n \epsilon_1^n + [n-1]_q \,\mathrm{id}_n \right)$$

and

$$\Psi_1^n(x_0^{n-1}) = \sum_{i=1}^{n-1} [i]_q \delta_i^n \epsilon_1^n.$$

The sum of these two expressions is

$$[n]_q[n-1]_q \operatorname{id}_n + [n-1]_q\delta_i^n\epsilon_1^n + \sum_{i=1}^{n-2}([n]_q[n-i-1]_q + [i]_q)\delta_i^n\epsilon_1^n.$$
$$= [n-1]_q\left(\sum_{i=0}^{n-1}[n-i]_q\delta_i^n\epsilon_1^n + [n]_q \operatorname{id}_n\right),$$

due to Lemma 6.3, so $x_1^n = \sum_{i=0}^{n-1}[n-i]_q\delta_i^n\epsilon_1^n + [n]_q \operatorname{id}_n$ which Proposition 3.2 states is annihilated by all the generators. □

6.2 Inductive Computation for General k

In this section, we prove that x_k^n, as defined inductively by $[n-k]_q x_k^n = [n]_q\Theta_k^n(x_k^{n-1}) + \Psi_k^n(x_{k-1}^{n-1})$, is annihilated by the actions of all the generators. The vectors $\Theta_k^n(x_k^{n-1})$ and $\Psi_k^n(x_{k-1}^{n-1})$ are both annihilated by the actions of $e_1, \ldots, e_{n-2}$, so by Proposition 6.1, a linear combination of them must be annihilated by e_{n-1}. The action of e_{n-1} is defined by the action of ϵ_{n-1}^n, so we consider this action for the sake of simplicity. Computing the coefficients of ϵ_k^n allows us to obtain the linear combination, so it is sufficient to compute $\operatorname{coeff}_{\epsilon_k^n}(\epsilon_{n-1}^n\Theta_k^n(x_k^{n-1}))$ and $\operatorname{coeff}_{\epsilon_k^n}(\epsilon_{n-1}^n\Psi_k^n(x_{k-1}^{n-1}))$ and show that the coefficient of ϵ_k^n of $[n]_q\epsilon_{n-1}^n\Theta_k^n(x_k^{n-1}) + \epsilon_{n-1}^n\Psi_k^n(x_{k-1}^{n-1})$ is equal to 0.

We reduce the problem of computing $\operatorname{coeff}_{\epsilon_k^n}(\epsilon_{n-1}^n\Theta_k^n(x_k^{n-1}))$ and $\operatorname{coeff}_{\epsilon_k^n}(\epsilon_{n-1}^n\Psi_k^n(x_{k-1}^{n-1}))$ to computing the coefficient of id_{n-i} in x_j^{n-i} in Proposition 6.5, Proposition 6.6, and Lemma 6.7. We then compute the coefficient of id_{n-i} in x_j^{n-i} in Lemma 6.8, leading to Theorem 6.9, an inductive formula for computing $\varphi_k^n(W_n^n)$.

First, we reduce the problem of computing $\operatorname{coeff}_{\epsilon_k^n}(\epsilon_{n-1}^n\Theta_k^n(x_k^{n-1}))$.

Proposition 6.5. *We have that* $\operatorname{coeff}_{\epsilon_k^n}(\epsilon_{n-1}\Theta_k^n(x_k^{n-1})) = \operatorname{coeff}_{\operatorname{id}_{n-2}}(x_{k-1}^{n-2})$.

Proof. By the inductive hypothesis, $[n-k-1]_q x_k^{n-1} = [n-1]_q\Theta_k^{n-1}(x_k^{n-2}) + \Psi_k^{n-1}(x_{k-1}^{n-2})$. We must compute the coefficient of ϵ_k^n of $\frac{1}{[n-k-1]_q}(\epsilon_{n-1}^n\Theta_k^n\Theta_k^{n-1}(x_k^{n-2}) + \epsilon_{n-1}\Theta_k^n\Psi_k^{n-1}(x_{k-1}^{n-2}))$. Since Θ_i^j is a map adding a straight link to the right, $\Theta_k^n\Theta_k^{n-1}(x_k^{n-2})$ is a diagram with a straight link on both the $(n-1)$th and nth vertices, so the action of ϵ_{n-1} sends it to 0. Therefore, we only need to compute the coefficient of ϵ_k^n in $\frac{1}{[n-k-1]_q}\epsilon_{n-1}\Theta_k^n\Psi_k^{n-1}(x_{k-1}^{n-2})$.

Note that $\epsilon_{n-1}(D)$ is zero if D has two straight links to its right. As the action of Θ_k^n adds a straight link to the right of a diagram, $\Psi_k^{n-1}(x_{k-1}^{n-2})$ must have a simple link between the $(n-2)$th and $(n-1)$th vertices on the top row.

We show that $\Psi_k^{n-1}(D)$ for $D \in C_{k-1}^{n-2}$ has ϵ_k^n as a term if and only if D is the identity diagram on $n-2$ vertices. Note that by Theorem 5.6, we obtain that

$$\Psi_k^{n-1}(D) = D\delta_k^{n-1}\epsilon_k^{n-1} + [2]_q D\delta_{k+1}^{n-1}\epsilon_k^{n-1} + \cdots + [n-k-1]_q D\delta_{n-2}^{n-1}\epsilon_k^{n-1}.$$

If D is a nontrivial diagram in C_{k-1}^{n-2}, we only need to consider $D\delta_{n-2}^{n-1}\epsilon_k^{n-1}$ as it is the only possibility where a simple link connecting the $(n-2)$th and $(n-1)$th vertices on the top line appears. Furthermore, D must also have a simple link between the $(k-1)$th and kth vertices. In the edge case where $k \geq n-2$, there will no longer be a link connecting the $(n-1)$th and $(n-2)$th vertices in $D\delta_{n-2}^{n-1}\epsilon_k^{n-1}$, so it will be sent to 0. If $k < n-2$, the link connecting the $(k-1)$th and kth vertices in D will create a second quasi-simple link on the bottom line in $D\delta_{n-2}^{n-1}\epsilon_k^{n-1}$. Therefore, in this case, $\operatorname{coeff}_{\epsilon_k}(\Psi_k^{n-1}(D)) = 0$.

If $D = \mathrm{id}_{n-2}$, recall that by Definition 5.3, $\Psi_k^{n-1}(D) = \delta_k^{n-1}\epsilon_k^{n-1} + [2]_q\delta_{k+1}^{n-1}\epsilon_k^{n-1} + \cdots + [n-k-1]_q\delta_{n-2}^{n-1}\epsilon_k^{n-1}$. In this case, as $\epsilon_{n-1}^n\delta_i^n\epsilon_k^n = \epsilon_k^n$ when $k \leq i \leq n-2$ implies that $i = n-2$, we obtain that $\operatorname{coeff}_{\epsilon_k^n}(\epsilon_{n-1}^n\Theta_k^n\Psi_k^{n-1}(\mathrm{id}_{n-2})) = [n-k-1]_q$. Therefore,

$$\operatorname{coeff}_{\epsilon_k}(\epsilon_{n-1}\Theta_k^n(x_k^{n-1})) = \frac{1}{[n-k-1]_q}\operatorname{coeff}_{\epsilon_k^n}(\epsilon_{n-1}^n\Theta_k^n\Psi_k^{n-1}(x_{k-1}^{n-2})) = \operatorname{coeff}_{\mathrm{id}_{n-2}}(x_{k-1}^{n-2}). \qquad \square$$

Similarly to the above proposition, in the following proposition, we reduce the problem of computing $\operatorname{coeff}_{\epsilon_k^n}(\epsilon_{n-1}^n\Psi_k^n(x_{k-1}^{n-1}))$.

Proposition 6.6. *We have the recurrence relation*

$$\operatorname{coeff}_{\epsilon_k^n}(\epsilon_{n-1}^n\Psi_k^n(x_{k-1}^{n-1})) = -[n-k+1]\operatorname{coeff}_{\mathrm{id}_{n-1}}(x_{k-1}^{n-1}) + \operatorname{coeff}_{\delta_{n-2}^{n-1}\epsilon_{k-1}^{n-1}}(x_{k-1}^{n-1}).$$

Proof. We show that if $\Psi_k^n(D)$ has a term that is ϵ_k^n, $D \in C_{k-1}^{n-1}$ must be either id_{n-1} or $\delta_{n-2}^{n-1}\epsilon_{k-1}^{n-1}$. Recall that by Theorem 5.6, we obtain that $\Psi_k^n(D) = D\delta_k^n\epsilon_k^n + [2]_q D\delta_{k+1}^n\epsilon_k^n + \cdots + [n-k]_q D\delta_{n-1}^n\epsilon_k^{n-1}$.

In the case where $D = \mathrm{id}_{n-1}$, only $\epsilon_{n-1}^n\delta_{n-2}^n\epsilon_k^n$ and $\epsilon_{n-1}^n\delta_{n-1}^n\epsilon_k^n$ create a coefficient of ϵ_k^n. The coefficient of ϵ_k^n in $\Psi_k^n(\mathrm{id}_{n-1})$ is $[n-k-1]-[2][n-k] = -[n-k+1]$. In the case where D is not the identity, for the coefficient of ϵ_k^n to be nonzero, D must be equal to $\delta_{n-2}^{n-1}\epsilon_{k-1}^{n-1}$ as otherwise there will be a simple link on the top line. If this is the case, the only term where a coefficient of ϵ_k^n appears is $\delta_{n-2}^{n-1}\epsilon_{k-1}^{n-1}\delta_k^n\epsilon_k^n$. In this case, the coefficient of ϵ_k^n in $\Psi_k^n(\delta_{n-2}^{n-1}\epsilon_{k-1}^{n-1})$ is 1. Combining this information, we obtain this proposition. $\square$

We further reduce the original problem, proving that computing the coefficient of the identity diagram is sufficient to compute $\mathrm{coeff}_{\delta^{n-1}_{n-2}\epsilon^{n-1}_{k-1}}(x^{n-1}_{k-1})$.

Lemma 6.7. *We have the relation* $\mathrm{coeff}_{\delta^{n-1}_{n-2}\epsilon^{n-1}_{k-1}}(x^{n-1}_{k-1}) = \mathrm{coeff}_{\mathrm{id}_{n-2}}(x^{n-2}_{k-2})$.

Proof. As $[n-1-k]_q x^{n-1}_{k-1} = [n-1]_q \Theta^{n-1}_{k-1}(x^{n-2}_{k-1}) + \Psi^{n-1}_{k-1}(x^{n-2}_{k-2})$ and each component of $\Theta^{n-1}_{k-1}(x^{n-2}_{k-1})$ has a rightmost straight link as Θ^{n-1}_{k-1} is a map that adds a straight link to the right, we obtain that $\mathrm{coeff}_{\delta^{n-1}_{n-2}\epsilon^{n-1}_{k-1}}(x^{n-1}_{k-1}) = \mathrm{coeff}_{\delta^{n-1}_{n-2}\epsilon^{n-1}_{k-1}}(\Psi^{n-1}_{k-1}(x^{n-2}_{k-2}))$. Similarly to the proof of Proposition 6.5, $\Psi^{n-1}_{k-1}(D)$ has a term of $\delta^{n-1}_{n-2}\epsilon^{n-1}_{k-1}$ if and only if $D = \mathrm{id}_{n-2}$. Therefore,

$$\mathrm{coeff}_{\delta^{n-1}_{n-2}\epsilon^{n-1}_{k-1}}(x^{n-1}_{k-1}) = [n-k]_q \mathrm{coeff}_{\mathrm{id}_{n-2}}(x^{n-2}_{k-2}).$$ □

Define $[a]_q! = [a]_q[a-1]_q \dots [1]_q$ for integer a. Furthermore, let $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$.

We can compute the coefficient of the identity diagram inductively by using the recurrence found in Lemma 6.7.

Lemma 6.8. *We obtain the formula* $\mathrm{coeff}_{\mathrm{id}_{n-1}}(x^{n-1}_{k-1}) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$.

Proof. Since $x^{n-1}_{k-1} = \frac{[n-1]_q}{[n-k]_q}\Theta^{n-1}_{k-1}(x^{n-2}_{k-1}) + \frac{1}{[n-k]!}\Psi^{n-1}_{k-1}(x^{n-2}_{k-2})$ by the inductive hypothesis and every term in $\Psi^{n-1}_{k-1}(x^{n-2}_{k-2})$ is nontrivial, we conclude that

$$\mathrm{coeff}_{\mathrm{id}_{n-1}}(x^{n-1}_{k-1}) = \frac{[n-1]_q}{[n-k]_q}\mathrm{coeff}_{\mathrm{id}_{n-1}}(\Theta^{n-1}_{k-1}(x^{n-2}_{k-1})).$$

Repeating this process, we obtain that

$$\mathrm{coeff}_{\mathrm{id}_{n-1}}(x^{n-1}_{k-1}) = \frac{[n-1]_q}{[n-k]_q} \dots \frac{[k]_q}{[1]_q} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$ □

Combining all the results in this section, we can prove the following theorem, one of the main results of our paper.

Theorem 6.9. *The vector x^n_k in $(\mathbb{C}^2)^{\otimes n}_k$, defined by the inductive formula $[n-k]_q x^n_k = [n]_q\Theta^n_k(x^{n-1}_k) + \Psi^n_k(x^{n-1}_{k-1})$, is annihilated by the actions of all e_is.*

Proof. We must show that the action of ϵ^n_{n-1} on $[n]_q\Theta^n_k(x^{n-1}_k) + \Psi^n_k(x^{n-1}_{k-1})$ is equal to 0. By Proposition 6.1, this is equivalent to proving that the coefficient of ϵ^n_k of $[n]_q\epsilon^n_{n-1}\Theta^n_k(x^{n-1}_k) + \epsilon^n_{n-1}\Psi^n_k(x^{n-1}_{k-1})$ is equal to 0.

By Proposition 6.5, Proposition 6.6, Lemma 6.7, and Lemma 6.8, we have that

$$\begin{aligned}\frac{\mathrm{coeff}_{\epsilon_k^n}(\epsilon_{n-1}^n\Psi_k^n(x_{k-1}^{n-1}))}{\mathrm{coeff}_{\epsilon_k^n}(\epsilon_{n-1}^n\Theta_k^n(x_k^{n-1}))} &= \frac{-[n-k+1]_q\mathrm{coeff}_{\mathrm{id}_{n-1}}(x_{k-1}^{n-1})+\mathrm{coeff}_{\delta_{n-2}^{n-1}\epsilon_{k-1}^{n-1}}(x_{k-1}^{n-1})}{\mathrm{coeff}_{\mathrm{id}_{n-2}}(x_{k-1}^{n-2})}\\ &= \frac{-[n-k+1]_q\begin{bmatrix}n-1\\k-1\end{bmatrix}_q+\begin{bmatrix}n-2\\k-2\end{bmatrix}_q}{\begin{bmatrix}n-2\\k-1\end{bmatrix}_q}\\ &= \frac{-[n-k+1]_q[n-1]_q+[k-1]_q}{[n-k]_q}\\ &= -[n]_q.\end{aligned}$$

This implies that $[n]_q\Theta_k^n(x_k^{n-1})+\Psi_k^n(x_{k-1}^{n-1})$ is a vector that is annihilated by all generators in $(\mathbb{C}^2)_k^{\otimes n}$, proving the theorem. □

Theorem 6.9 gives an inductive procedure for the computation of $\varphi_k^n(W_n^n)$, while the results in Section 4, specifically Theorem 4.9 allow us to reduce the computation of $\varphi_k^n(W_{n-2i}^n)$ to the computation of $\varphi_k^{n-2i}(W_{n-2i}^{n-2i})$. These two cases cover all of $W_n^n\oplus\cdots\oplus W_{n-2k}^n$, allowing us to explicitly compute the irreducible decomposition of the spin representation into Specht modules.

Example 6.10. We explicitly compute x_2^4. We have that $[2]_qx_2^4=[4]\Theta_2^4(x_2^3)+\Psi_2^4(x_1^3)$. By Proposition 3.2, $\Theta_2^4(x_2^3)=[1]_q\delta_1^4\epsilon_2^4+[2]_q\delta_2^4\epsilon_2^4+[3]_q\,\mathrm{id}_4$ and $x_1^3=[1]_q\delta_2^3\epsilon_1^3+[2]_q\delta_1^3\epsilon_1^3+[3]_q\,\mathrm{id}_3$. Therefore,

$$\Psi_2^4(x_1^3)=[1]_q(\delta_2^4\epsilon_2^4+[2]_q\delta_2^4\delta_2^2\epsilon_2^2\epsilon_2^4)+[2]_q(\delta_1^4\epsilon_2^4+\delta_1^4\delta_3^4\epsilon_2^2\epsilon_2^4)+[3]_q(\delta_2^4\epsilon_2^4+[2]_q\delta_3^4\epsilon_2^4).$$

From these, we obtain that

$$\begin{aligned}[2]_qx_2^4 =&[4]_q([1]_q\delta_1^4\epsilon_2^4+[2]_q\delta_2^4\epsilon_2^4+[3]_q\,\mathrm{id}_4)\\ &+[1]_q(\delta_2^4\epsilon_2^4+[2]_q\delta_2^4\delta_2^2\epsilon_2^2\epsilon_2^4)+[2]_q(\delta_1^4\epsilon_2^4+\delta_1^4\delta_3^4\epsilon_2^2\epsilon_2^4)+[3]_q(\delta_2^4\epsilon_2^4+[2]_q\delta_3^4\epsilon_2^4)\\ =&[4]_q[3]_q\,\mathrm{id}_4+([4]_q+[2]_q)\delta_1^4\epsilon_2^4+([4]_q[2]_q+1+[3]_q)\delta_2^4\epsilon_2^4\\ &+[3]_q[2]_q\delta_3^4\epsilon_2^4+[2]_q\delta_2^4\delta_2^2\epsilon_2^2\epsilon_2^4+[2]_q\delta_1^4\delta_3^4\epsilon_2^2\epsilon_2^4.\end{aligned}$$

Therefore,

$$x_2^4=([5]_q+1)\,\mathrm{id}_4+[3]_q\delta_1^4\epsilon_2^4+([4]_q+[2]_q)\delta_2^4\epsilon_2^4+[3]_q\delta_3^4\epsilon_2^4+\delta_2^4\delta_2^2\epsilon_2^2\epsilon_2^4+\delta_1^4\delta_3^4\epsilon_2^2\epsilon_2^4.$$

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