REPRESENTATION-THEORETIC BACKGROUND

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Abstract. We briefly review combinatorics and algebraic representation of reductive groups: root systems, Weyl groups and (extended) affine Weyl groups, weight spaces, and classification of irreducible representations in terms of highest weights. We define the Langlands dual group. We review p-adic groups and their (smooth) representations, including: Hecke algebras attached to compact open subgroups, relation between representations and Hecke modules, classical Satake isomorphism and its proof (sketch). If time permits, we discuss the unramified local Langlands correspondence.

Contents

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Part 1. Algebraic groups and their algebraic representations

We follow [Hum75].

1. Algebraic groups

Throughout the talk, let k be a field. Let G be a connected split reductive group over k. We do not review what that means. We fix a split maximal torus $T \subseteq G$.

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Example 1.1. A typical example is $G = \mathrm{GL}_n$. In this case,

$$T = \left\{ \begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{pmatrix} \in G \right\}$$

satisfies the definition. Note $\mathrm{GL}_1 = \mathbb{G}_{\mathrm{m}}$.

1.1. **Weyl groups.** The *Weyl group* W of G with respect to T is defined as

$$W \coloneqq N_G(T)/T.$$

This is a finite constant group scheme over k.

Example 1.2. If $G = \mathrm{GL}_n$ and T is as above, then W is isomorphic to the symmetric group $\mathfrak{S}_n$. The isomorphism is given by associating a permutation $\sigma \in \mathfrak{S}_n$ with the class of

$$\begin{pmatrix} 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \end{pmatrix}$$

where the first column has 1 at the $\sigma(1)$-th row, the second column has 1 at the $\sigma(2)$-th row, and so on.

The set of Borel subgroups of G containing T is acted on by W simply transitively.

Example 1.3. If $G = \mathrm{GL}_n$ and T is as above, then the set of Borel subgroups of G containing T is identified with the set of complete flags of k^n that are stable under the action of T. Especially, the set of regular upper triangular matrices containing T is a Borel subgroup. Also, the set of regular lower triangular matrices containing T is another Borel subgroup.

When $G = \mathrm{SL}_n$, and T is the set of diagonal matrices in SL_n, then the Borel subgroups containing T are the intersections of the above Borel subgroups with SL_n.

1.2. **Root systems.** Set $\mathfrak{g}, \mathfrak{t}$ to be the Lie algebras of G and T. Define

$$X^*(T) \coloneqq \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}}), \quad X_*(T) \coloneqq \mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, T).$$

The adjoint action of T on $\mathfrak{g}$ decomposes $\mathfrak{g}$ into weight spaces:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{0 \neq \alpha \in X^*(T)} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha \coloneqq \{ v \in \mathfrak{g} \mid \mathrm{Ad}(t)(v) = \alpha(t)v, \quad \forall t \in T \}.$$

Let $\Phi(G, T)$ be the set of $0 \neq \alpha \in X^*(T)$ such that $\mathfrak{g}_\alpha \neq 0$. It receives the action of W.

Example 1.4. If $G = \mathrm{GL}_n$ and T is as above, then

$$\Phi(G, T) = \{ e_i - e_j \mid 1 \leqq i, j \leqq n, i \neq j \},$$

where $e_i \in X^*(T)$ is the standard basis. The action of $W \simeq \mathfrak{S}_n$ on $\Phi(G, T)$ is given by

$$\sigma(e_i - e_j) = e_{\sigma(i)} - e_{\sigma(j)}.$$

The same description holds when $G = \mathrm{SL}_n$ and T is the set of diagonal matrices in SL_n.

Proposition 1.5. *If* $\alpha \in \Phi(G,T)$*, then* $\dim \mathfrak{g}_\alpha = 1$.

Definition 1.6. An element $\alpha \in \Phi(G,T)$ is called a *root*. Fix a Borel subgroup B containing T. A root α is called *positive* (with respect to B) if $\mathfrak{g}_\alpha \subseteq \mathrm{Lie}(B)$. Let $\Phi^+(G,T)$ be the set of positive roots. Let $\Delta(G,T) \subseteq \Phi^+(G,T)$ be the set of *simple roots*, i.e., the positive roots that cannot be written as a sum of two or more positive roots.

Proposition 1.7.
(1) *We have* $\Phi(G,T) = \Phi^+(G,T) \sqcup (-\Phi^+(G,T))$.
(2) *Every positive root is written as a sum of simple roots.*
(3) *The set of simple roots is a basis of the* $\mathbb{Q}$*-vector space* $X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 1.8. A reduced root datum is a tuple $(X^*, \Phi, X_*, \Phi^\vee, \Phi \to \Phi^\vee, \langle *, * \rangle \colon X^* \times X_* \to \mathbb{Z})$ consisting of

- free abelian groups X^*, X_* of finite rank,
- a perfect pairing $\langle *, * \rangle \colon X^* \times X_* \to \mathbb{Z}$,
- finite subsets $\Phi \subseteq X^*$ and $\Phi^\vee \subseteq X_*$ with a bijection $\Phi \to \Phi^\vee, \alpha \mapsto \alpha^\vee$,

that satisfy the following conditions.

(1) For every $\alpha \in \Phi$, we have $\langle \alpha, \alpha^\vee \rangle = 2$.
(2) For every $\alpha \in \Phi$, the reflection $s_\alpha \colon X^* \to X^*$ defined by
$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$$
preserves Φ.
(3) For every $\alpha \in \Phi$, the reflection $s_{\alpha^\vee} \colon X_* \to X_*$ defined by
$$s_{\alpha^\vee}(\mu) = \mu - \langle \alpha, \mu \rangle \alpha^\vee$$
preserves $\Phi^\vee$.
(4) For each $\alpha \in \Phi$, we have $\mathbb{Q}\alpha \cap X^* = \{\pm\alpha\} \subset X^*$.

Proposition 1.9. *Let* $(X^*, \Phi, X_*, \Phi^\vee, \dots)$ *be a reduced root datum. Then for any* $\mu \in \Phi^\vee$*, we have* $\mathbb{Q}\mu \cap X_* = \{\pm\mu\} \subset X_*$.

Corollary 1.10. *Let* $(X^*, \Phi, X_*, \Phi^\vee, \dots)$ *be a reduced root datum. Then* $(X_*, \Phi^\vee, X^*, \Phi, \dots)$ *is also a reduced root datum.*

Theorem 1.11.
(1) *Let* G, T *as before. Then there exist a unique bijection* $\Phi^\vee(G,T) \to \Phi^\vee(G,T) \subset X_*(T)$ *that makes* $(X^*(T), \Phi(G,T), X_*(T), \Phi^\vee(G,T), \dots)$ *a reduced root datum.*
(2) *The isomorphism classes of the pair of a connected split reductive group over* k *and its split maximal torus are in bijection with the isomorphism classes of reduced root data.*

All of the reduced root datum corresponding to (G,T) receives the action of W.

Example 1.12. Let $G = \mathrm{SL}_2$. Identify its diagonal torus T with $\mathbb{G}_{\mathrm{m}}$ by
$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t.$$
Then $\Phi(G,T) = \{\pm 2\} \subset \mathbb{Z} = \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}})$. We have $2^\vee = 1 \in \mathbb{Z} = \mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, T)$.

Definition 1.13. Let G, T as before. We define the split reductive group $\widehat{G}$ over $\mathbb{C}$ and its split maximal torus $\widehat{T}$ so that they correspond to $(X_*(T), \Phi^\vee(G,T), X^*(T), \Phi(G,T), \dots)$.

1.3. **Extended affine Weyl groups.** We follow [WAR, Lecture 1]. The *extended affine Weyl group* of G with respect to T is defined as $W \ltimes X_*(T)$. It acts on $X_*(T)$ by

$$(w, \mu) \cdot \mu' = w(\mu' + \mu).$$

2. The highest weight theory

Take a finite dimensional vector space V over k. Let $G \to \mathrm{GL}(V)$ be an algebraic representation. As in the case of T acting on $\mathfrak{g}$, we have

$$V = \bigoplus_{\alpha \in X^*(T)} V_\alpha,$$

where

$$V_\alpha \coloneqq \{v \in V \mid t \cdot v = \alpha(t)v, \quad \forall t \in T\}.$$

An element $\alpha \in X^*(T)$ with $V_\alpha \neq 0$ is called a *weight* of the representation.

Fix a Borel subgroup B containing T.

Definition 2.1. A character $\alpha \in X^*(T)$ is called *dominant* (with respect to B) if $\alpha - w(\alpha) \in \mathbb{N}\Delta(G, T)$ for all w in the Weyl group.

Theorem 2.2. *(1) Let V be an algebraic representation of G. There exists a unique weight $\alpha_V \in X^*(T)$ such that for any weight β of V, we have $\alpha_V - \beta \in \mathbb{N}\Delta(G, T)$. We have $\dim V_{\alpha_V} = 1$.*

(2) The isomorphism classes of irreducible algebraic representations of G are in bijection with the set of dominant characters of T by $V \mapsto \alpha_V$.

The character α_V is called the *highest weight* of V.

Example 2.3. Let $G = \mathrm{SL}_2$. Identify the diagonal torus T with $\mathbb{G}_\mathrm{m}$ as before. Let B be the Borel subgroup of the upper triangular matrices. Then a character $\alpha \in X^*(T) = \mathbb{Z}$ is dominant if and only if $\alpha \geqq 0$. Assume that the characteristic of k is 0. The irreducible algebraic representations of G corresponding to $n \in \mathbb{N}$ is $\mathrm{Sym}^n \mathrm{std}$, where $\mathrm{std}\colon G \to \mathrm{GL}_2$ is the standard inclusion. In positive characteristic, the corresponding irreducible representation is the subrepresentation of the same construction.

Part 2. p-adic groups and their representations

We follow [Car79].

3. p-adic groups

We reuse the notation in Part 1. Let $k = F$ be a non-archimedean local field, i.e., a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Let $\mathcal{O}_F$ be its ring of integers, ϖ a uniformizer, and $\mathbb{F}_q = \mathcal{O}_F/(\varpi)$ its residue field. Let $v_F\colon F \twoheadrightarrow \mathbb{Z}$ be the valuation. Set $|*| \coloneqq q^{-v_F(*)}$ to be the absolute value on F.

Let H be a locally profinite group. Namely, H is a totally disconnected locally compact Hausdorff group, e.g., $G(F)$. Equivaletly, H is a Hausdorff topological group, and has a basis of neighborhoods of the identity consisting of compact open subgroups.

3.1. **Smooth representations.** Let V be a complex vector space. A representation (π, V) of H is called *smooth* if for any $v \in V$, there exists an open subgroup $K \subseteq H$ such that $\pi(k)(v) = v$ for all $k \in K$. As in the case of locally compact topological groups or Lie groups, there are notions of induced representations, compact inductions, Frobenius reciprocity, Schur's lemma, and so on.

3.2. **Hecke algebras.** Fix a left Haar measure μ on H. Let $K \subseteq H$ be a compact open subgroup. The *Hecke algebra* $\mathcal{H}(H, K)$ is defined as the set of compactly supported K-biinvariant complex-valued functions on H. It becomes a ring by the convolution product

$$(f * g)(x) = \int_H f(h')g(h'^{-1}x)d\mu(h') = \int_H f(xh')g(h'^{-1})d\mu(h'), \quad \forall f, g \in \mathcal{H}(H, K), x \in H,$$

with the unit element $\mathbb{1}_K/\mu(K)$. Let

$$\mathcal{H}(H) := \varinjlim_{K \subseteq H} \mathcal{H}(H, K),$$

and call it the *Hecke algebra* of H. This equals the set of compactly supported locally constant complex-valued functions on H. The convolution product extends to $\mathcal{H}(H)$.

Example 3.1. Let $H = \mathrm{GL}_n(F)$ and $K = \mathrm{GL}_n(\mathcal{O}_F)$. The theory of invariant factors tells us that $K\backslash H/K$ is in bijection with the set of non-increasing sequences of integers $(a_1, a_2, \dots, a_n)$ by

$$(a_1, a_2, \dots, a_n) \mapsto K \begin{pmatrix} \varpi^{a_1} & 0 & \cdots & 0 \\ 0 & \varpi^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varpi^{a_n} \end{pmatrix} K.$$

Let (π, V) be a smooth representation of H. Then $\mathcal{H}(H)$ acts on V by

$$\pi(f)v := \int_H f(h)\pi(h)(v)d\mu(h), \quad \forall f \in \mathcal{H}(H), v \in V.$$

For a compact open subgroup $K \subseteq H$, the action of $\mathcal{H}(H, K)$ preserves $V^K := \{v \in V \mid \pi(k)(v) = v, \forall k \in K\}$.

Proposition 3.2. *(1) The category of smooth representations of H is equivalent to the category of $\mathcal{H}(H)$-modules M such that $\mathcal{H}(H)M = M$.*

(2) Let (π, V) be a nonzero smooth representation of H. Then V is irreducible if and only if V^K is eigher 0 or a simple $\mathcal{H}(H, K)$-module for any compact open subgroup $K \subseteq H$.

(3) Let $K \subseteq H$ be a compact open subgroup. Then the functor $V \mapsto V^K$ from the category of smooth representations of H to the category of $\mathcal{H}(H, K)$-modules is exact. The functor gives a bijection between the isomorphism classes of irreducible smooth representations (π, V) of H with $V^K \neq 0$ and the isomorphism classes of simple $\mathcal{H}(H, K)$-modules.

4. Satake isomorphism

Let G be a split connected reductive group over F. Our G extends to a split connected reductive group scheme over $\mathcal{O}_F$ (also denoted by G). Let $K = G(\mathcal{O}_F)$, which is a maximal compact subgroup of $H := G(F)$. The *spherical Hecke algebra* of H is defined as $\mathcal{H}(H, K)$. The goal here is to describe the structure of $\mathcal{H}(H, K)$ as well as the corresponding representation theory.

Definition 4.1. A smooth representation (π, V) of H is called *unramified* if $V^K \neq 0$.

Take T, a split maximal torus in G. Fix a Borel subgroup B containing T. Let N be the maximal unipotent subgroup of B.

Example 4.2. Let $G = \mathrm{GL}_n$. Suppose that B is the Borel subgroup of upper triangular matrices. Then N is the subgroup of upper triangular matrices with 1's on the diagonal.

Normalize the left Haar measures on $\Gamma \in \{H, T(F), N(F), K\}$ so that $\Gamma \cap K$ has volume 1. Define $\mathrm{ord}_T \colon T(F) \to X_*(T)$ by carrying $t \in T(F)$ to the element of $X_*(T) = \mathrm{Hom}(X^*(T), \mathbb{Z})$ that sends $\alpha \in X^*(T)$ to $v_F(\alpha(t)) \in \mathbb{Z}$. It is surjective with kernel $T(F) \cap K$. It induces an isomorphism

$$\mathcal{H}(T(F), T(F) \cap K) \xrightarrow{\sim} \mathbb{C}[X_*(T)].$$

Define the *Satake transform* $\mathcal{H}(H,K) \to \mathcal{H}(T(F), T(F) \cap K)$ by carrying $f \in \mathcal{H}(H,K)$ to

$$Sf \colon t \mapsto \delta(t)^{1/2} \int_{N(F)} f(tn)dn = \delta(t)^{-1/2} \int_{N(F)} f(nt)dn, \quad \forall t \in T(F),$$

where $\delta(t) := |\det(\mathrm{Ad}(t)|_{\mathrm{Lie}\, N})|$.

Theorem 4.3. *The Satake transform induces an isomorphism*

$$\mathcal{H}(H,K) \xrightarrow{\sim} \mathbb{C}[X_*(T)]^W$$

of $\mathbb{C}$-algebras, where the action of W on $\mathbb{C}[X_*(T)]$ is induced by the action of W on $X_*(T)$.

Corollary 4.4. *The ring $\mathcal{H}(H,K)$ is commutative. Its simple modules are one-dimensinal.*

Example 4.5. Let $G = \mathrm{GL}_2$. Then we have

$$\mathcal{H}(G(F), G(\mathcal{O}_F)) \simeq \mathbb{C}[X_*(T)]^{\mathfrak{S}_2} \simeq \mathbb{C}[e_1 + e_2, (e_1e_2)^{\pm 1}],$$

where e_i is the standard basis of $X_*(T)$.

On the other hand, we have seen that $G(\mathcal{O}_F)\backslash G(F)/G(\mathcal{O}_F)$ is in bijection with the set of non-increasing pairs of integers (a_1, a_2). Let $\mathbb{1}_{a_1,a_2}$ be the characteristic function of the double coset corresponding to (a_1, a_2). Then $\{\mathbb{1}_{a_1,a_2} \mid a_1 \geqq a_2\}$ is a basis of $\mathcal{H}(G(F), G(\mathcal{O}_F))$.

We compute $S\mathbb{1}_{1,0}$ and $S\mathbb{1}_{1,1}$. For the first one, we have

$$\begin{aligned} S\mathbb{1}_{1,0}(\mathrm{diag}(t_1,t_2)) &= \delta(\mathrm{diag}(t_1,t_2))^{-1/2} \int_{N(F)} \mathbb{1}_{1,0}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right) dx \\ &= \left|\frac{t_1}{t_2}\right|^{-1/2} \int_{N(F)} \mathbb{1}_{1,0}\left(\begin{pmatrix} t_1 & t_2x \\ 0 & t_2 \end{pmatrix}\right) dx, \end{aligned}$$

where

$$\mathrm{diag}(t_1,t_2) := \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}.$$

The matrix in the integrand of the most right side belongs to the double coset corresponding to $(1,0)$ if and only if $v_F(t_1t_2) = 1, (t_1, t_2x, t_2) = \mathcal{O}_F$. This happens exactly when either $(v_F(t_1), v_F(t_2)) = (1,0)$ and $x \in \mathcal{O}_F$, or $(v_F(t_1), v_F(t_2)) = (0,1)$ and $x \in \varpi^{-1}\mathcal{O}_F^\times$. Thus

$$\begin{aligned} S\mathbb{1}_{1,0}(\mathrm{diag}(t_1,t_2)) =& (q^{-1})^{-1/2} \cdot 1 \cdot \mathbb{1}_{\mathrm{diag}(\varpi,1)(T(F)\cap K)}(\mathrm{diag}(t_1,t_2)) \\ &+ q^{-1/2} \cdot q \cdot \mathbb{1}_{\mathrm{diag}(1,\varpi)(T(F)\cap K)}(\mathrm{diag}(t_1,t_2)) \\ =& q^{1/2}(e_1 + e_2). \end{aligned}$$

We move on to $S\mathbb{1}_{1,1}$. The matrix

$$\begin{pmatrix} t_1 & t_2x \\ 0 & t_2 \end{pmatrix}$$

belongs to the double coset corresponding to $(1,1)$ if and only if $v_F(t_1t_2) = 2, (t_1, t_2x, t_2) = \varpi\mathcal{O}_F$. This is equivalent to $v_F(t_1) = v_F(t_2) = 1$ and $x \in \mathcal{O}_F$. Therefore, as in the case of $S\mathbb{1}_{1,0}$, we have

$$S\mathbb{1}_{1,1}(\mathrm{diag}(t_1,t_2)) = 1^{-1/2} \cdot 1 \cdot \mathbb{1}_{\mathrm{diag}(\varpi,\varpi)(T(F)\cap K)}(\mathrm{diag}(t_1,t_2)) = e_1e_2.$$

Definition 4.6. Any irreducible unramified smooth representation of H corresponds to a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[X_*(T)]^W = \mathcal{H}(H,K) \to \mathbb{C}$, namely a point of $\widehat{T}(\mathbb{C})/W$, where as before $\widehat{T} = \operatorname{Hom}(X_*(T), \mathbb{C}^\times) = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times$ is the complex torus dual to T. Call this point the *Satake parameter* of the representation.

The Satake parameter can also be thought of as a semisimple element of $\widehat{G}(\mathbb{C})$ considered up to conjugacy. This is the content of the *unramified local Langlands correspondence*.

Proof. We sketch the construction of the inverse of the Satake isomorphism. Take $\chi \in \widehat{T}(\mathbb{C})/W$. By lifting it to an elememt of $\widehat{T}(\mathbb{C})$, we obtain a character $\chi\colon B(F) \to T(F) \xrightarrow{\operatorname{ord}_T} X_*(T) \xrightarrow{\chi} \mathbb{C}^\times$ and a $\mathbb{C}$-algebra homomorphism $\chi\colon \mathbb{C}[X_*(T)] \to \mathbb{C}$. We also have the character $\delta\colon B(F) \to T(F) \xrightarrow{\delta} q^{\mathbb{Z}}$. Consider the normalized induced representation $\operatorname{Ind}_{B(F)}^H \delta^{1/2}\chi$, the set of locally constant functions $\phi\colon H \to \mathbb{C}$ such that

$$\phi(bh) = \delta^{1/2}(b)\chi(b)\phi(h), \quad \forall b \in B(F), h \in H.$$

The space $(\operatorname{Ind}_{B(F)}^H \delta^{1/2}\chi)^K$ is one-dimensional because $H = B(F)K$. Take a nonzero vector ϕ in it. Also take $f \in \mathcal{H}(H,K)$. We compute $\pi(f)\phi$.

$$\begin{aligned}
\pi(f)\phi(1) &= \int_H f(h)\phi(h)d\mu(h) \\
&= \int_{T(F)} \int_{N(F)} \int_K f(tnk)\phi(tnk)d\mu(k)d\mu(n)d\mu(t) \\
&= \int_{T(F)} \int_{N(F)} f(tn)\phi(tn)d\mu(n)d\mu(t) \\
&= \phi(1) \int_{T(F)} \delta^{1/2}(t)\chi(t) \int_{N(F)} f(tn)d\mu(n)d\mu(t) \\
&= \phi(1) \int_{T(F)} \chi(t)Sf(t)d\mu(t) \\
&= \phi(1)\chi(Sf).
\end{aligned}$$

The induction has the finite length, so it has a unique unramified irreducible subquotient corresponding to χ by the above computation. □

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