WAKIMOTO SHEAVES AND WAKIMOTO FILTRATION

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In this talk, we lift some of our knowledge about the Iwahori Hecke algebra from Naomi's talk to the Iwahori Hecke category. More precisely, we have constructed a large commutative subalgebra generated by Bernstein elements and used it to study the center of Iwahori Hecke algebra. We'll lift the Bernstein elements to Wakimoto sheaves and use them to study central sheaves. We'll mainly follow the reference [1].

1. Wakimoto sheaves

Consider local function field $F = \mathbb{C}((t))$ and its ring of integers $O = C[\![t]\!]$. Fix a semisimple, simply-connected algebraic group G over $\mathbb{C}$ and a theory D of constructible sheaves[1](de Rham, Betti or étale) with $D(*) = k$-mod for a field k. Fix a maximal torus and a Borel subgroup $T \subset B \subset G$. We use the letter I both for the Iwahori subgroup of $G(F)$ associated to B and the corresponding subgroup scheme of LG. Denote by $W := N_G(T)(F)/T(O)$ the extended affine Weyl group of G, which has two natural subgroups: the finite Weyl group $W_f := N_G(T)(\mathbb{C})/T(\mathbb{C})$ and the cocharacter lattice $X_*(T) = T(F)/T(O)$. The Borel subgroup determines the cone $X_*(T)^+$ of dominant cocharacters and a set S of Coxeter generators of W. The length function on the Coxeter group W will be denoted by l, whose restriction on $X_*(T)^+$ recovers the function $\langle 2\rho, -\rangle$ for ρ the half sum of positive roots.

We have the affine flag variety $\mathrm{Fl}_G := LG/I$ and Hecke stack $\mathcal{H}_I$ of Iwahori level. We'll mainly be concerned with objects in the category $D(\mathcal{H}_I)$, which will be equivalently seen as I-equivariant constructible sheaves on Fl_G. The category $D(\mathcal{H}_I)$ is equipped with the perverse t-structure and a monoidal structure $*$ given by convolution.

Consider Schubert cells on the affine flag variety. The Bruhat decomposition $G(F) = \bigsqcup_{w\in W} IwI$ gives us a stratification $\mathrm{Fl}_G = \bigsqcup_{w\in W} \mathrm{Fl}_{G,w}$, with the w-cell $\mathrm{Fl}_{G,w}$ isomorphic to the affine space of dimension $l(w)$. Therefore we have affine immersions $j_w \colon \mathcal{H}_{I,w} := I\backslash \mathrm{Fl}_{G,w} \hookrightarrow \mathcal{H}_I$ for $w \in W$.

Definition 1.1. We define the *standard objects* $\Delta_w := j_{w,!}\underline{k}[l(w)]$ and the *costandard objects* $\nabla_w := j_{w,*}\underline{k}[l(w)]$ in $D(\mathcal{H}_I)$ for $w \in W$.

Lemma 1.2.

1. *For $w_1, w_2 \in W$ with $l(w_1) + l(w_2) = l(w_1w_2)$, there are canonical isomorphisms $\Delta_{w_1} * \Delta_{w_2} \cong \Delta_{w_1w_2}, \nabla_{w_1} * \nabla_{w_2} \cong \nabla_{w_1w_2}$.*
2. *For $w \in W$, we have canonical isomorphisms $\Delta_w * \nabla_{w^{-1}} \cong \mathbb{1} \cong \nabla_{w^{-1}} * \Delta_w$.*
3. *For $w_1, w_2 \in W$, the convolution products $\Delta_{w_1} * \nabla_{w_2}, \nabla_{w_2} * \Delta_{w_1}$ are perverse. In particular, the objects Δ_w, ∇_w are perverse for $w \in W$.*

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[1]Since we will encounter algebraic stacks that are not safe in the sense of [2], it's helpful to keep the following example in mind: the category of sheaves on $B\,\mathbb{G}_m$ is equivalent to $H^*(\mathbb{G}_m)$-mod, while the full subcategory of constructible sheaves is equivalent to $\mathrm{Perf}(H^*(B\,\mathbb{G}_m))$.

Proof.

(1) This is because the convolution map $\mathrm{Fl}_{G,w_1} \tilde{\times} \mathrm{Fl}_{G,w_2} \to \mathrm{Fl}_{G,w_1 w_2}$ is an isomorphism when $l(w_1) + l(w_2) = l(w_1 w_2)$.
(2) By writing w as a word and apply (1), we may reduce to the situation when $w = s$ is a simple reflection. Now $\mathrm{Fl}_{G,\leq s}$ is isomorphic to $\mathbb{P}^1$ and the lemma is a simple computation with Radon transform.
(3) Observe that the maps $m_1\colon \mathrm{Fl}_{G,w_1} \tilde{\times} \mathrm{Fl}_G \to \mathrm{Fl}_G, m_2\colon \mathrm{Fl}_G \tilde{\times} \mathrm{Fl}_{G,w_2} \to \mathrm{Fl}_G$, both restricted from the convolution map, are affine since Schubert cells are affine spaces. Write the sheaf $\Delta_{w_1} * \nabla_{w_2}$ either as $m_{1,!}(k[l(w_1)] \tilde{\boxtimes} \nabla_{w_2})$ or $m_{2,*}(\Delta_{w_1} \tilde{\boxtimes} k[l(w_2)])$, the lemma follows from half t-exactness of pushforward along affine maps.

□

Definition 1.3. For $\mu \in X_*(T)$, we define *Wakimoto sheaves*

$$J_\mu := \operatorname{colim} \nabla_{\mu_1} * \Delta_{-\mu_2},$$

where we take the filtered colimit indexed by $\mu = \mu_1 - \mu_2$ with $\mu_1, \mu_2 \in X_*(T)^+$ and transition maps isomorphisms provided by Lemma 1.2.

Remark 1.4. The construction geometrizes the Bernstein elements Θ_μ from Naomi's talk.

Lemma 1.5. *For $\mu, \nu \in X_*(T)$, $\mathrm{RHom}(J_\mu, J_\nu)$ vanishes unless $\nu \leq \mu$.*

Proof. We can assume both μ, ν are dominant by convolving with some Wakimoto sheaf J_λ with $\lambda \gg 0$. Then the lemma follows from the fact that $j_\nu^* \nabla_\mu$ vanishes unless $\nu \leq \mu$, since ∇_μ is supported on $\mathrm{Fl}_{G,\leq \mu}$. □

The lemma says $\{J_\mu\}_{\mu \in X_*(T)}$ looks like a semi-orthogonal decomposition of the category they generate. We'll exploit this in the next section

It's natural to combine these objects into a monoidal functor $J\colon \mathrm{Rep}(\check{T})^{\omega,\heartsuit} \to D(\mathcal{H}_I)^\heartsuit, \mu \mapsto J_\mu$. Inspecting the proof of Lemma 1.2, if we use the projections $\mathcal{H}_{I,w} \to BI \to BT$ to identify the category $D(\mathcal{H}_{I,w})$ with $D(BT)$, the functor J can be further enhanced to

$$\mathcal{J} = \oplus_\mu \mathcal{J}_\mu\colon \mathrm{Rep}(\check{T})^\omega \otimes D(BT) \to D(\mathcal{H}_I)$$

by setting $J_\mu = j_{\mu,*}[\langle 2\rho, \mu\rangle]$ for $\mu \in X_*(T)^+$ and requiring t-exact monoidal with respect to the usual t-structure and tensor monoidal structure on $D(BT)$. We observe each $\mathcal{J}_\mu$ is fully faithful since $\mathcal{J}$ is monoidal and J_μ is $*$-invertible.

2. Wakimoto filtration

Define the category Wak of Wakimoto filtered objects as the full subcategory of $D(\mathcal{H}_I)$ generated by the essential image of J[2] under finite limits and colimits. Define $\mathrm{Wak}^\heartsuit$ as the full subcategory of Wak consisting of objects that admits a filtration with graded pieces in the essential image of J. Since J is monoidal, both categories $\mathrm{Wak}^\heartsuit \subset \mathrm{Wak}$ are closed under convolution product, thus equipped with monoidal structure.

For each object $\mathcal{F} \in \mathrm{Wak}$, we'll construct a canonical *Wakimoto filtration* from Lemma 1.5. We index the filtration by the poset $\mathcal{K}$ of subsets $K \subset X_*(T)$ with finitely

[2] Note the definition doesn't change if we use the enhanced functor $\mathcal{J}$ instead, because $D(BT)$ is generated by the constant sheaf.

many maximal elements such that $\mu \le \nu, \nu \in K$ implies $\mu \in K$ for $\mu, \nu \in X_*(T)$, with partial order given by inclusion. For such subsets K, we can similarly form Wak_K, the full subcategory generated by $J_\mu, \mu \in K$ under finite limits and colimits, and we have $\mathrm{Wak} = \mathrm{colim}_{\mathcal{K}} \mathrm{Wak}_K$.

For a maximal element $\mu \in K$, the fully faithful functor $\mathcal{J}_\mu \colon D(BT) \to \mathrm{Wak}_K$ admits a left adjoint. Indeed, we can assume μ is dominant by convolving with some Wakimoto sheaf J_ν for $\nu \gg 0$ and then the left adjoint $\mathcal{J}_\mu^L$ is nothing but $j_\mu^*[-\langle 2\rho, \mu\rangle]$. We claim the functor $\mathcal{F} \mapsto \mathrm{fib}(\mathcal{F} \to \mathcal{J}_\mu \mathcal{J}_\mu^L \mathcal{F})$ gives a right adjoint of the inclusion $\mathrm{Wak}_{K\setminus\{\mu\}} \subset \mathrm{Wak}_K$. The functor kills J_μ and restricts to identity on $\mathrm{Wak}_{K\setminus\{\mu\}}$ since $J_\mu^L(\mathrm{Wak}_{K\setminus\{\mu\}}) = 0$ by Lemma 1.5. Because Wak_K is generated by J_μ and $\mathrm{Wak}_{K\setminus\{\mu\}}$, the functor takes image in $\mathrm{Wak}_{K\setminus\{\mu\}}$. It is clear that this is the right adjoint functor by again applying Lemma 1.5

By composing and taking colimit, the inclusion $\mathrm{Wak}_K \subset \mathrm{Wak}$ admits a right adjoint $\mathcal{F} \mapsto \mathcal{F}_K$, which gives us the promised Wakimoto filtration after forgetting back to Wak. The fact that J is monoidal and Lemma 1.5 implies the Wakimoto filtration is monoidal with respect to Day convolution.

We form the graded pieces $\mathrm{Gr}_\mu \colon \mathrm{Wak} \to \mathrm{Wak}, \mathcal{F} \mapsto \mathrm{cofib}(\mathcal{F}_{<\mu} \to \mathcal{F}_{\le\mu})$ and combine them to a monoidal functor $\mathrm{Gr} = \oplus_\mu \mathrm{Gr}_\mu$. By arguments above, the image of Gr_μ lands in the essential image of $\mathcal{J}_\mu$, so we can lift Gr_μ to $\mathrm{gr}_\mu \colon \mathrm{Wak} \to D(BT)$ and Gr to a monoidal functor $\mathrm{gr} \colon \mathrm{Wak} \to \mathrm{Rep}(\check{T}) \otimes D(BT)$. For $\mathcal{F} \in \mathrm{Wak}^\heartsuit$, $\mathrm{gr}\mathcal{F}$ takes value in $\mathrm{Rep}(\check{T})^\heartsuit$, viewed as an object of the target of gr via the constant sheaf.

We'll apply the formalism above to the central sheaves, so it's necessary to show the central sheaves are Wakimoto filtered. This follows from the criterion given below. We need some notations: for $\mathcal{F} \in D(\mathcal{H}_I)$, define $\mathrm{supp}(\mathcal{F}) := \{w \in W \colon j_w^* \mathcal{F} \ne 0\}$ and $\mathrm{cosupp}(\mathcal{F}) := \{w \in W \colon j_w^! \mathcal{F} \ne 0\}$.

Theorem 2.1.

(1) *For any $\mathcal{F} \in D(\mathcal{H}_I)$, the following are equivalent:*
   (a) *$\mathcal{F}$ is Wakimoto filtered, i.e. $\mathcal{F} \in \mathrm{Wak}$,*
   (b) $\mathrm{supp}(J_\mu * \mathcal{F}) \subset -X_*(T)^+, \mu \ll 0$,
   (c) $\mathrm{cosupp}(J_\mu * \mathcal{F}) \subset X_*(T)^+, \mu \gg 0$.

(2) *The conditions in (1) are satisfied if we are given isomorphisms*

$$J_\mu * \mathcal{F} \simeq \mathcal{F} * J_\mu, \forall \mu \in X_*(T).$$

(3) *If $\mathcal{F} \in \mathrm{Wak}$ is convolution exact, i.e. convolution with $\mathcal{F}$ is perverse t-exact, then $F \in \mathrm{Wak}^\heartsuit$.*

Proof.

(1) Note that $\mathrm{supp}(J_\mu) = \mathrm{supp}(\Delta_\mu) = \{\mu\}$ for μ anti-dominant and $\mathrm{cosupp}(J_\mu) = \mathrm{cosupp}(\nabla_\mu) = \{\mu\}$ for μ dominant. Thus (a) implies (b),(c) since only finitely many J_μ can appear in $\mathcal{F} \in \mathrm{Wak}$. Conversely, if $\mathrm{supp}(\mathcal{F}) \subset -X_*(T)^+$, then $\mathcal{F}$ is filtered by standard objects $\Delta_\mu, \mu \in -X_*(T)^+$ by excision, thus Wakimoto filtered. Similarly (c) implies (a).

(2) We need the following observation.

Claim 1. For any $\mathcal{F} \in D(\mathcal{H}_I)$, there exists a finite subset $S_\mathcal{F} \subset W$, such that $\mathrm{cosupp}(\nabla_w * \mathcal{F}) \subset w \cdot S_\mathcal{F}, \mathrm{cosupp}(\mathcal{F} * \nabla_w) \subset S_\mathcal{F} \cdot w$ holds for any $w \in W$.

Assuming the claim, then we know $\mathrm{cosupp}(J_\mu * \mathcal{F}) \subset \mu \cdot S_{\mathcal{F}} \cap S_{\mathcal{F}} \cdot \mu$ for $\mu \in X_*(T)^+$. Take $\mu \gg 0$, then equality $\mu \cdot S_{\mathcal{F}} \cap S_{\mathcal{F}} \cdot \mu = \mu \cdot (S_{\mathcal{F}} \cap X_*(T))$ holds and the latter set is a subset of $X_*(T)^+$ by finiteness.

The claim is proven by induction. It suffices to treat convolution on the left, by taking unions. By dévissage, we're reduced to the case $\mathcal{F} = \nabla_v, v \in W$. In the base case $l(v) = 0$, we simply set $S_{\mathcal{F}} = \{v\}$. If $l(v) > 0$, we may take a simple reflection $s \in W$ such that $l(sv) < l(v)$ and write $\nabla_v = \nabla_s * \nabla_{sv}$. By Bruhat decomposition, $\nabla_w * \nabla_s$ is in the full subcategory generated by ∇_w and ∇_{ws} under finite limits and colimits. As a result, the set $S_{\nabla_v} = S_{\nabla_{sv}} \cup (s \cdot S_{\nabla_{sv}})$ satisfy our requirement.

(3) Without loss of generality, we may assume $\mathcal{F}$ is perverse and $\mathrm{cosupp}(\mathcal{F}) \subset X_*(T)^+$. Take a maximal element $\mu \in \mathrm{cosupp}(\mathcal{F})$, then $\mathrm{Gr}_\mu \mathcal{F} = j_{\mu,*} j_\mu^* \mathcal{F}$ is perverse. Replace $\mathcal{F}$ by the fiber of $\mathcal{F} \to \mathrm{Gr}_\mu \mathcal{F}$ and an induction on the size of $\mathrm{cosupp}(\mathcal{F})$ yields the result.

□

In particular, we have shown the central sheaves are objects of $\mathrm{Wak}^{\heartsuit}$.

3. Cohomology and central sheaf

We denote the central functor by $Z\colon \mathrm{Rep}(\check{G})^{\omega,\heartsuit} \to \mathrm{Wak}^{\heartsuit}$.The goal of this section is to prove the following theorem.

Theorem 3.1. *The monoidal functor* $\mathrm{gr} \circ Z\colon \operatorname{Rep} \check{G}^{\omega,\heartsuit} \to \operatorname{Rep} \check{T}^{\omega,\heartsuit}$ *is isomorphic to the functor of restricting representations.*

The approach we'll take is using cohomology supported on semi-infinite orbits to separate different $\check{T}$-weights.

Let $\mathbb{G}_{\mathrm{m}}$ act on Fl_G though a strictly dominant cocharacter, so that we can run the constructions in hyperbolic localization. Based on the fact below, we can (set theoretically) identify the $\mathbb{G}_{\mathrm{m}}$-fixed points Fl_G^0 with W and the retractor[3] Fl_G^+ with the disjoint union of semi-infinite orbits $\bigsqcup_{w \in W} S_w, S_w := LUwI/I$.

Lemma 3.2 (Iwasawa decomposition)**.** *There's a bijection of sets*

$$W = N_G(T)(F)/T(O) \xrightarrow{\simeq} U(F)\backslash G(F)/I.$$

We have a correspondence $\mathrm{Fl}_G \xleftarrow{p} \mathrm{Fl}_G^+ \xrightarrow{q} \mathrm{Fl}_G^0$. Define constant term functor as $\mathrm{CT_B} = q_* p^!$. Concretely, the value of $\mathrm{CT_B}(\mathcal{F})$ at $w \in W$ is computed by $R\Gamma(S_w, \mathcal{F}|^!_{S_w})$. We can also consider $\mathrm{CT}_{B'}$ for other Borel subgroups B' containing T since Lemma 3.2 still holds. In order to apply constant term to Wakimoto sheaves, we need to understand its compatibility with convolution.

Lemma 3.3. *For* $\mathcal{F}_1, \mathcal{F}_2 \in D(\mathcal{H}_I)$ *and* $w \in W$*, we have*

$$\mathrm{CT_B}(\mathcal{F}_1 * \mathcal{F}_2)_w \cong \oplus_{w_1 w_2 = w} \mathrm{CT_B}(\mathcal{F}_1)_{w_1} \otimes \mathrm{CT}_{\mathrm{Ad}_{w_{1,f}^{-1}}(B)}(\mathcal{F}_2)_{w_2}.$$

Here $w_{1,f} \in W_f$ *stands for the finite part of* w_1*. In particular, we can ignore the twist if* $\mathrm{CT_B}(\mathcal{F}_1)$ *only supports on* $X_*(T)$*.*

Proof. We make the following observation.

[3]For a space X with $\mathbb{G}_{\mathrm{m}}$-action, the retractor X^+ parametrize $\mathbb{G}_{\mathrm{m}}$-equivariant maps from $\mathbb{A}^1$ to X, where $\mathbb{G}_{\mathrm{m}}$ acts on $\mathbb{A}^1$ by scaling. We can similarly define repeller X^- by using inverse scaling action on $\mathbb{A}^1$ instead.

Observation 3.4 (Hyperbolic localization is compatible with proper pushforward)**.** Let X, Y be schemes with $\mathbb{G}_m$ action and $f\colon Y \to X$ be a proper $\mathbb{G}_m$-equivariant map. Then we get the following commutative diagram.

$$\begin{array}{ccccc} Y & \xleftarrow{p_Y} & Y^+ & \xrightarrow{q_Y} & Y^0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^+} & & \downarrow{\scriptstyle f^0} \\ X & \xleftarrow{p_X} & X^+ & \xrightarrow{q_X} & X^0 \end{array}$$

Notice the left square is Cartesian by valuative criterion, so we get

$$q_{X,*}p_X^! f_* \cong q_{X,*}f_*^+ p_Y^! \cong f_*^0 q_{Y,*}p_Y^!.$$

Apply the observation to $m\colon \mathrm{Fl}_G \tilde{\times} \mathrm{Fl}_G \to \mathrm{Fl}_G$. Under the isomorphism

$$(pr_1, m)\colon \mathrm{Fl}_G \tilde{\times} \mathrm{Fl}_G \simeq \mathrm{Fl}_G \times \mathrm{Fl}_G,$$

we identify $(\mathrm{Fl}_G \tilde{\times} \mathrm{Fl}_G)^0$ with $(\mathrm{Fl}_G \times \mathrm{Fl}_G)^0 = W \times W$. The preimage of (w_1, w_1w_2) in $(\mathrm{Fl}_G \tilde{\times} \mathrm{Fl}_G)^+$ can be computed as

$$\begin{aligned} S_{w_1} \tilde{\times} S_{w_2} &:= (pr_1, m)^{-1}(S_{w_1} \times S_{w_1w_2}) \\ &= LU \times^{LU \cap w_1 I w_1^{-1}} w_1^{-1} LU w_1 \cdot w_2 I/I \end{aligned}$$

The space $w_1^{-1}LUw_1 \cdot w_2 I/I$ is a semi-infinite orbit indexed by w_2 associated to the Borel subgroup $\mathrm{Ad}_{w_{1,f}^{-1}}(B)$. The group $LU \cap w_1 I w_1^{-1}$ is pro-unipotent, thus has no effect on cohomology. So we get

$$R\Gamma(S_{w_1} \tilde{\times} S_{w_2}, \mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2|^!_{S_{w_1} \tilde{\times} S_{w_2}}) = \mathrm{CT}_{\mathrm{B}}(\mathcal{F}_1)_{w_1} \otimes \mathrm{CT}_{\mathrm{Ad}_{w_{1,f}^{-1}}(B)}(\mathcal{F}_2)_{w_2}.$$

We conclude by combining the computation with Observation 3.4, noticing our constructible sheaves are always supported on finitely many strata. □

Now we compute the constant term functor on Wakimoto sheaves.

Corollary 3.5. $\mathrm{CT}_{\mathrm{B}}(J_\mu)_w \cong k[\langle 2\rho, \mu\rangle]$ *if* $w = \mu$*, and zero otherwise.*

Proof. For μ dominant, we simply notice L^+U acts on $\mathrm{Fl}_{G,\mu}$ transitively by computing with Lie algebra. This implies the case for general μ by Lemma 3.3. □

As a result of the corollary, for a Wakimoto filtered perverse sheaf $\mathcal{F} \in \mathrm{Wak}^\heartsuit$, the constant term $\mathrm{CT}_{\mathrm{B}}(\mathcal{F})$ is supported on $X_*(T)$ and we can then identify $\mathrm{gr}\mathcal{F}$ with $\oplus_\mu \mathrm{CT}_{\mathrm{B}}(\mathcal{F})_\mu[-\langle 2\rho, \mu\rangle] \cdot \mu$. This is an isomorphism of monoidal functors from $\mathrm{Wak}^\heartsuit$ to $\mathrm{Rep}(\check{T})$ by Lemma 3.3.

Finally, we treat central sheaves. Recall if we denote $\pi\colon \mathrm{Fl}_G \to \mathrm{Gr}_G$ the natural projection, then we have isomorphism $\pi_* \circ Z \cong id$. Apply Observation 3.4 to π, we make the following computation[4] for $V \in \mathrm{Rep}\, \check{G}^{\omega,\heartsuit}$

$$\begin{aligned} \mathrm{gr}Z(V) &\cong \oplus_\mu \mathrm{CT}_{\mathrm{B}}(Z(V))_\mu[-\langle 2\rho, \mu\rangle] \cdot \mu \\ &\cong \oplus_\mu (\pi^0_* \mathrm{CT}_{\mathrm{B}}(Z(V)))_\mu[-\langle 2\rho, \mu\rangle] \cdot \mu \\ &\cong \oplus_\mu \mathrm{CT}_{\mathrm{B}}(\pi_* Z(V))_\mu[-\langle 2\rho, \mu\rangle] \cdot \mu \\ &\cong \oplus_\mu \mathrm{CT}_{\mathrm{B}}(\mathrm{IC}_V)_\mu[-\langle 2\rho, \mu\rangle] \cdot \mu \\ &\cong \oplus_\mu V(\mu) \cdot \mu = V|_{\check{T}}. \end{aligned}$$

[4]In the last line, one may prefer to write $V(w_0 \cdot \mu)$ for w_0 the longest element in W_f. This is about how geometric Satake is normalized.

Moreover, all the isomorphisms are monoidal in V by Lemma 3.3 and centrality of central sheaves. This finishes the proof of Theorem 3.1.

References

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