

DERIVED GEOMETRIC SATAKE EQUIVALENCE

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1. PRELIMINARIES AND STATEMENT OF THE MAIN THEOREM

1.1. Review of geometric Satake. We work over \mathbf{C} and sheaves are taken with respect to the analytic topology. Étale sheaves should work equally well. Let G be a reductive group over \mathbf{C} . We additionally assume that G is simply connected. Occasionally, we comment on how to adjust the statements when G is not simply connected.

Then

$$LG : R \mapsto G(R((t))), \quad L^+G : R \mapsto G(R[[t]])$$

are fpqc sheaves; moreover, L^+G is an affine scheme, and LG is an ind-affine ind-scheme over \mathbf{C} . The affine Grassmannian is

$$\mathrm{Gr} = \mathrm{Gr}_G := LG/L^+G.$$

All quotients are stacky. The fpqc sheaf Gr is represented by an ind-projective ind-scheme. It carries a left L^+G -action, and there is a reasonable abelian category of L^+G -equivariant perverse sheaves (with coefficients in \mathbf{C}) on Gr , or equivalently, a reasonable abelian category of perverse sheaves on $L^+G \backslash \mathrm{Gr}$, which moreover carries a reasonable symmetric monoidal structure given by convolution product and certain commutativity constraints, making the hypercohomology functor

$$\mathcal{H}^\bullet : \mathrm{Perv}_{L^+G}(\mathrm{Gr}) \rightarrow \mathrm{Vect}_{\mathbf{C}}$$

a fiber functor, which identifies $\mathrm{Perv}_{L^+G}(\mathrm{Gr})$ with the category of representations of some affine algebraic group via the Tannakian formalism. Choose $T \subset B \subset G$. The category $\mathrm{Perv}_{L^+G}(\mathrm{Gr})$ is semisimple and its simple objects are IC_λ where λ is a dominant coweight and IC_λ is the intersection complex on $\mathrm{Gr}_{\leq \lambda}$, i.e., the intermediate extension of $\underline{\mathbf{C}}[[2\rho, \lambda]]$ on Gr_λ , where 2ρ is the sum of all positive roots.

Let \check{G} denote the Langlands dual group of G . Its isomorphism classes of finite irreducible representations are in canonical bijection with dominant weights of \check{G} , which are dominant coweights of G . Then the geometric Satake equivalence asserts that there exists an equivalence of symmetric monoidal categories

$$\mathrm{Perv}_{L^+G}(\mathrm{Gr}) \simeq \mathrm{Rep}(\check{G})$$

sending $\mathrm{IC}_\lambda \mapsto V_\lambda$ to the corresponding highest weight representation; moreover, the equivalence sends \mathcal{H}^\bullet to the forget functor $\mathrm{Rep}(\check{G}) \rightarrow \mathrm{Vect}_{\mathbf{C}}$.

The goal of this talk is to understand the derived counterpart of the above statement, as in [BF07]. Namely, we wish to understand the derived (∞ -)category $\mathcal{D}_{c, L^+G}^b(\mathrm{Gr}, \mathbf{C}) = \mathcal{D}_{L^+G}(\mathrm{Gr})$ in terms of the dual group. We will always work with the ∞ -category of bounded constructible equivariant sheaves, and we shall simplify the notation by just writing $\mathcal{D}_{L^+G}(\mathrm{Gr})$ from now on.

1.2. The loop rotation action. Bezrukavnikov-Finkelberg [BF07] also gave a description of the loop rotation equivariant derived category by the Harish-Chandra bimodules for the dual Lie algebra. We have to first explain what these mean. We start by explaining the loop rotation action.

There is a natural action of \mathbf{G}_m on LG given by

$$R^\times \times G(R((t))) \rightarrow G(R((t))), \quad (r, g) \mapsto g \circ r$$

where we view $g : \mathrm{Spec}(R((t))) \rightarrow G$ and $r : \mathrm{Spec}(R((t))) \rightarrow \mathrm{Spec}(R((t)))$ given by $t \mapsto rt$. Similarly, \mathbf{G}_m acts on L^+G and the actions are compatible with $L^+G \hookrightarrow LG$.

It thus makes sense to form the semidirect product $L^+G \rtimes \mathbf{G}_m$ where explicitly the multiplication map is given by

$$(g, r) \times (g', r') = (gr(g), rr').$$

It is easy to verify that the left \mathbf{G}_m -action and the left L^+G -action on Gr together gives a $(L^+G \rtimes \mathbf{G}_m)$ -action on Gr .

The \mathbf{G}_m -action preserves all Schubert cells. To see this, recall that $\mathrm{Gr}_\lambda = L^+G \cdot \lambda(t)$ and $r \in R^\times$ sends it to

$$r(L^+G \cdot \lambda(t)) = L^+G \cdot \lambda(rt) = L^+G \cdot \lambda(t)$$

where the last equality follows from the fact that $\lambda(r) \in T(R) \subset G(R) \subset L^+G(R)$. As in Longke's talk, this guarantees that there is a stratification on $(L^+G \rtimes \mathbf{G}_m) \backslash \mathrm{Gr}$ by $\mathbf{X}_*(T)^+$ which can be used to define a perverse t -structure on $\mathcal{D}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr})$ with heart $\mathrm{Perv}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr})$.

Now we wish to understand the relationship between $\mathrm{Perv}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr})$ with the Satake category $\mathrm{Perv}_{L^+G}(\mathrm{Gr})$ and the corresponding relationship between derived categories. There are pullback functors

$$\mathcal{D}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}) \rightarrow \mathcal{D}_{L^+G}(\mathrm{Gr}) \rightarrow \mathcal{D}(\mathrm{Gr}).$$

The restriction of the above functors to the abelian subcategory of perverse sheaves yield functors

$$\mathrm{Perv}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}) \rightarrow \mathrm{Perv}_{L^+G}(\mathrm{Gr}) \rightarrow \mathrm{Perv}(\mathrm{Gr}).$$

The last map and the composition are both full faithful, by [Zhu16, Lemma A.1.2]. The image of the last map is the semisimple full subcategory generated by finite sum of IC_λ 's. Since the \mathbf{G}_m -action preserves all Schubert cells, the intermediate extension of the trivial local systems on the λ -indexed strata of $(L^+G \rtimes \mathbf{G}_m) \backslash \mathrm{Gr}$ pulls back to IC_λ on Gr . Thus, the composition and the last map has the same essential image. Therefore we have an equivalence

$$\mathrm{Perv}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}) \simeq \mathrm{Perv}_{L^+G}(\mathrm{Gr}).$$

In particular the geometric Satake equivalence extends to

$$\mathrm{Perv}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}) \simeq \mathrm{Rep}(\check{G}).$$

In view of this fact, it seems also reasonable to interpret $\mathcal{D}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr})$ in terms of the dual group and call this derived geometric Satake. This was indeed what Bezrukavnikov-Finkelberg did.

1.3. Kostant slice. Our choice of (T, B) gives a preferred choice of (\check{T}, \check{B}) inside \check{G} . We also have preferred choices of Cartan and Borel $\check{\mathfrak{t}} \subset \check{\mathfrak{b}}_+ \subset \check{\mathfrak{g}}$. Let $\check{\mathfrak{b}}_-$ be the opposite Borel. Let $\check{\mathfrak{n}}_+$ and $\check{\mathfrak{n}}_-$ be the nilpotent radicals of two opposite Borels. Let W be the Weyl group of G and \check{G} , naturally identified.

Let $e, h, f \in \check{\mathfrak{g}}$ be a principal \mathfrak{sl}_2 triple such that $f \in \check{\mathfrak{n}}_-$ and $e \in \check{\mathfrak{n}}_+$. By ‘‘principal’’ we mean that e is regular nilpotent. When this is the case, the existence of such an \mathfrak{sl}_2 triple is guaranteed by the Jacobson-Morozov theorem. It is a theorem of Kostant that the Kostant slice $e + \mathfrak{z}_{\check{\mathfrak{g}}}(f) \subset \check{\mathfrak{g}}$ has the property that the restriction of the GIT quotient map

$$e + \mathfrak{z}_{\check{\mathfrak{g}}}(f) \rightarrow \check{\mathfrak{g}}//G$$

is an isomorphism, i.e., the Kostant slice intersects with each semisimple (or regular) adjoint orbit exactly once. Moreover, it is a fact that \check{N}_- acts on $e + \check{\mathfrak{b}}_-$ and there is an isomorphism

$$e + \mathfrak{z}_{\check{\mathfrak{g}}}(f) \simeq (e + \check{\mathfrak{b}}_-)//\check{N}_-.$$

To see this map, as a consequence of the regularity of e , we have $\mathfrak{z}_{\check{\mathfrak{g}}}(f) \subset \check{\mathfrak{b}}_-$ and this induces a natural map from the left to the right.

1.4. Harish-Chandra bimodules. Let

$$U = U(\check{\mathfrak{g}}) := T(\check{\mathfrak{g}})/(x \otimes y - y \otimes x - [x, y])$$

be the universal enveloping algebra of $\check{\mathfrak{g}}$. Then U carries a natural PBW filtration where $F^n U(\check{\mathfrak{g}}) \subset U$ consists of linear combinations of products of at most n elements in $\check{\mathfrak{g}}$. The associated graded algebra is the symmetric power $\mathrm{gr}(U) = \mathrm{Sym}(\check{\mathfrak{g}})$.

let U_{\hbar} be an one-parameter deformation of U given by the Rees construction applied to the PBW filtration

$$U_{\hbar} := \bigoplus_{n=0}^{\infty} F^n U \cdot \hbar^n,$$

where \hbar is central. This is an associative $\mathbf{C}[\hbar]$ -algebra. Equivalently

$$U_{\hbar} := T(\check{\mathfrak{g}})[\hbar]/(x \otimes y - y \otimes x - \hbar[x, y]).$$

The natural isomorphism is given by $x \mapsto x\hbar$ from the second construction to the first. Let's not get into the details of proving this. We will use the second construction in this note.

The adjoint \check{G} -action on $\check{\mathfrak{g}}$ extends to a \check{G} -action on U preserving the filtered pieces and hence induces an action on U_{\hbar} .

Definition 1.1. A \hbar -Harish-Chandra bimodule M is a graded $U_{\hbar}^2 := U_{\hbar} \otimes_{\mathbf{C}[\hbar]} U_{\hbar} \simeq U_{\hbar} \otimes_{\mathbf{C}} U$ -module equipped with an algebraic action of \check{G}

$$\rho : \check{G} \times M \rightarrow M,$$

where M is here viewed an affine space over \mathbf{C} , such that

(i) the action map

$$U_{\hbar}^2 \otimes_{\mathbf{C}} M \rightarrow M$$

is \check{G} -equivariant, and

(ii) for $x \in \check{\mathfrak{g}}$, the actions of $(x \otimes 1 + 1 \otimes x) \in U_{\hbar}^2$ coincides with $\hbar \cdot d\rho(x)$. To be more elaborate, differentiating ρ we get an action

$$d\rho : \check{\mathfrak{g}} \times M \rightarrow M$$

and here by $\hbar \cdot d\rho(x)$ we mean composing the the action of $d\rho(x)$ on M with the action of $\hbar \in U_{\hbar}^2$.

Let \mathcal{HC}_{\hbar} denote the category of all \hbar -Harish-Chandra bimodules, and let $\mathcal{HC}_{\hbar}^{\text{fg}}$ denote the full subcategory of M that are finitely generated as U_{\hbar} -modules when restricting to the first factor (equivalently, the second factor) of U_{\hbar}^2 .

Remark 1.2. This looks complicated, so let's try to unwind it a bit. Restricting to the first factor of U_{\hbar}^2 , we see that M is a \check{G} -module with a \check{G} -equivariant U_{\hbar} -action. The action of the second factor in U_{\hbar}^2 is forced upon us by the second condition

$$(1 \otimes x)(m) = \hbar \cdot d\rho(x)(m) - xm.$$

We obtain a unique \check{G} -equivariant U_{\hbar}^2 -action on M , and hence we prove that the category of \hbar -Harish-Chandra bimodule is equivalent to the category of \check{G} -module with a \check{G} -equivariant U_{\hbar} -action. The latter is conceptually much simpler, while the notion of Harish-Chandra bimodules has a natural monoidal structure that is not readily visible here.

Remark 1.3. Consider the full subcategory of $\mathcal{HC}_{\hbar}^{\text{fg}}$ consisting of M on which \hbar acts by 0. Such M is a \check{G} -equivariant finitely generated $\text{Sym}(\check{\mathfrak{g}})$ -module, i.e., a \check{G} -equivariant coherent sheaf on $\check{\mathfrak{g}}^*$ viewed as an affine space. This subcategory is thus $\text{Coh}_{\check{G}}(\check{\mathfrak{g}}^*)$.

1.5. Deformation to the normal cone. We first need to review some basic constructions in algebraic geometry. Let X be a scheme and $Z \hookrightarrow X$ be a closed immersion cut out by an ideal sheaf \mathcal{J} . The sheaves of \mathcal{O}_X -algebras $\mathcal{O}_X[\hbar]$ and $\mathcal{O}_X[\hbar^{\pm 1}]$ have relative spectra over X are $X \times \mathbf{A}^1$ and $X \times \mathbf{G}_m$. Consider the sheaf (on X) of subalgebras in $\mathcal{O}_X[\hbar^{\pm 1}]$ generated by elements of the form $f\hbar^{-1}$ where $f \in \mathcal{J}$. More precisely, consider the sheaf

$$\mathcal{N} : U \mapsto \mathcal{O}_X(U)[\hbar] \langle f\hbar^{-1} : f \in \mathcal{J}(U) \rangle \subseteq \mathcal{O}_X(U)[\hbar^{\pm 1}],$$

where by $A\langle - \rangle$ we mean the A -subalgebra generated by elements in the bracket. This sheaf can also be constructed by some form of Rees construction

$$\mathcal{N} := \bigoplus_{n \in \mathbf{Z}} \mathcal{J}^n \hbar^{-n} \subset \mathcal{O}_X[\hbar^{\pm 1}],$$

where negative powers of \mathcal{J} are \mathcal{O}_X . Let $N_X Z$ denote the relative spectrum over X of \mathcal{N} . There is a natural inclusion map $\mathcal{O}_X[\hbar] \rightarrow \mathcal{N}$, inducing a natural map

$$N_X Z \rightarrow X \times \mathbf{A}^1$$

over X .

Restricting $N_X Z \rightarrow \mathbf{A}^1$ to \mathbf{G}_m , the parameter \hbar becomes invertible and hence $N_X Z|_{\mathbf{G}_m} \simeq X \times \mathbf{G}_m$. All fibers are just X at $\hbar \neq 0$. At $0 : * \rightarrow \mathbf{A}^1$, we have that

$$(N_X Z)_{\hbar=0} = \underline{\text{Spec}}_Z \left(\bigoplus_{n=0}^{\infty} \mathcal{J}^{n+1} / \mathcal{J}^n \right)$$

is the normal cone of Z in X . Thus outside $Z \times \{0\}$, $N_X Z \rightarrow X \times \mathbf{A}^1$ is an isomorphism. And we may think of $N_X Z$ as a one-parameter deformation of X into the normal cone $C_X Z$ at $\hbar = 0$. The normal cone $C_X Z$ is some sort of tubular neighborhood in algebraic geometry, and the deformation to the normal cone provides a flat family deforming X to $C_X Z$, making the analogy with differential topology useful in practice.

1.6. The Kostant functor κ_{\hbar} . We define the Kostant functor

$$\kappa_{\hbar} : \mathcal{H}\mathcal{C}_{\hbar}^{\text{fg}} \rightarrow \text{QCoh}_{\mathbf{G}_m}((\check{\mathfrak{t}}^* // W)^2 \times \mathbf{A}^1).$$

This functor will be useful when we try to describe what the hypercohomology functor corresponds to on the spectral side under derived geometric Satake.

We have a set of simple roots $\Delta = \{\alpha\}$ given by our preferred choice $(\check{\mathfrak{t}}, \check{\mathfrak{b}})$. For each simple root α , choose $f_{\alpha} \in \mathfrak{g}_{-\alpha}$ to be a generator of the root space corresponding to $-\alpha$. Sending $f_{\alpha} \mapsto 1$ for all $\alpha \in \Delta$ and killing all other roots induce a unique $\mathbf{C}[\hbar]$ -algebra morphism

$$\psi : U_{\hbar}(\check{\mathfrak{n}}_-) \rightarrow \mathbf{C}[\hbar].$$

As constructed ψ depends on the choices of f_{α} .

Write $U_{\hbar}^2(\check{\mathfrak{n}}_-) := U_{\hbar}(\check{\mathfrak{n}}_-) \otimes_{\mathbf{C}[\hbar]} U_{\hbar}(\check{\mathfrak{n}}_-) \subset U_{\hbar}^2$. Define

$$\psi^{(2)} : U_{\hbar}^2(\check{\mathfrak{n}}_-) \simeq U_{\hbar}(\check{\mathfrak{n}}_-) \otimes_{\mathbf{C}} U(\check{\mathfrak{n}}_-) \rightarrow \mathbf{C}[\hbar]$$

extending $\psi^{(2)}$ by tensoring ψ on the first factor and the trivial map on the second factor. By the trivial map we mean the map sending $x \mapsto 0$ for all $x \in \check{\mathfrak{n}}_-$. When viewed as a map

$$\psi^{(2)} : U_{\hbar}(\check{\mathfrak{n}}_-) \otimes_{\mathbf{C}[\hbar]} U_{\hbar}(\check{\mathfrak{n}}_-) \rightarrow \mathbf{C}[\hbar],$$

the restriction to the first factor is ψ and the restriction to the second factor is $-\psi$. Here by $-\psi$ we mean the composition

$$-\psi : U_{\hbar}(\check{\mathfrak{n}}_-) \xrightarrow{\psi} \mathbf{C}[\hbar] \xrightarrow{\hbar \mapsto -\hbar} \mathbf{C}[\hbar].$$

Equivalently, the map $-\psi$ is produced by the same procedure as above by insisting $f_{\alpha} \mapsto -1$.

Let $M \in \mathcal{H}\mathcal{C}_{\hbar}^{\text{fg}}$, define

$$\kappa_{\hbar}(M) := (M \otimes_{1 \otimes U_{\hbar}, -\psi} \mathbf{C}[\hbar])^{\check{N}_-},$$

where we view M as a U_{\hbar} -module by the action of the second factor $1 \otimes U_{\hbar} \subset U_{\hbar}^2$, and the \check{N}_- acts trivially on $\mathbf{C}[\hbar]$ and via the restriction of the \check{G} -action on M . Then the action of $1 \otimes Z(U_{\hbar})$ on M commutes with the \check{N}_- -action and passes to an action on $\kappa_{\hbar}(M)$.

In general, when M is both an A -module and a B -module for associative algebras A and B , and when $f : B \rightarrow C$ where C is commutative, M_C acquires a well-defined A action via

$$A \times (M \otimes_B C) \rightarrow M \otimes_B C, \quad (a, m \otimes c) \mapsto am \otimes c$$

if the A -action commutes with the B -action. In our situation, this means that the first factor $U_{\hbar} \otimes 1$ acts on $M \otimes \mathbf{C}[\hbar]$ defined above. Hence $Z(U_{\hbar}) \otimes 1$ also acts on $\kappa_{\hbar}(M)$. Together they give an action of the ‘‘Harish-Chandra center’’

$$Z(U_{\hbar}) \otimes_{\mathbf{C}[\hbar]} Z(U_{\hbar}) \simeq \Gamma((\check{\mathfrak{t}}^* // W)^2 \times \mathbf{A}^1, \mathcal{O})$$

on $\kappa_{\hbar}(M)$. To see why this is true, recall the Harish-Chandra isomorphism

$$Z(U) \simeq \text{Sym}(\check{\mathfrak{t}})^W \simeq \mathbf{C}[\check{\mathfrak{t}}^*]^W.$$

It’s easy to see then

$$Z(U_{\hbar}) \simeq Z(U)[\hbar] \simeq \Gamma(\check{\mathfrak{t}} // W \times \mathbf{A}^1, \mathcal{O}),$$

which explains the above isomorphism.

We defer the \mathbf{G}_m -equivariance part to the next subsection.

1.7. **Gradings via \hbar .** Recall we fixed an \mathfrak{sl}_2 -triple $e, h, f \in \check{\mathfrak{g}}$. Now h is well-defined up to conjugation. The adjoint action of h creates a grading on $\check{\mathfrak{g}}$ in terms of eigenvalues appearing

$$\check{\mathfrak{g}} = \bigoplus_{i \in \mathbf{Z}} \check{\mathfrak{g}}(i).$$

The adjoint action of h then acts on $U(\check{\mathfrak{g}})$. One observation used here is that in any associative algebra,

$$[h, uv] = [h, u]v + u[h, v].$$

For our purposes, give \hbar degree 2 in U_{\hbar} , using the Rees definition of U_{\hbar} , and this gives a grading on U_{\hbar} and hence on $Z(U_{\hbar})$. Thus $x \in \check{\mathfrak{g}}(i)$ has grading $2 + i$, as it is $x\hbar$ in the Rees construction definition. The reason for this degree 2 arrangement will soon be clear. To be clear, we always use the second (non-Rees) definition in the following and give $x \in \check{\mathfrak{g}}(i)$ grading $2 + i$.

Our principal \mathfrak{sl}_2 -triple, via $d\rho$, also produces an \mathfrak{sl}_2 -action on any $M \in \mathcal{HC}_{\hbar}$. The h -action then again produces a grading on M . The U_{\hbar}^2 -action on M respects the gradings.

To endow $\kappa_{\hbar}(M)$ with a natural grading, one may check that $\psi : U_{\hbar}(\check{\mathfrak{n}}_-) \rightarrow \mathbf{C}[\hbar]$ respects the gradings. It is a fact that the Harish-Chandra isomorphism

$$Z(U_{\hbar}) \simeq \Gamma(\check{\mathfrak{t}}^* \times \mathbf{A}^1, \mathcal{O})$$

is \mathbf{G}_m -equivariant where \mathbf{G}_m acts by multiplication by t^2 (instead of $t!$) on both factors on the right. Thus we prove that $\kappa_{\hbar}(M)$ is a \mathbf{G}_m -equivariant quasicoherent sheaf, and this finishes the definition of the Kostant functor.

Remark 1.4 (Interpretation by deformation to the normal cone). If M is free as a $\mathbf{C}[\hbar]$ -module. By the following elementary lemma, we know that $\kappa_{\hbar}(M)$ has a natural structure of a \mathbf{G}_m -equivariant quasicoherent sheaf on $N_{(\check{\mathfrak{t}}^*/W)^2} \Delta$.

Lemma 1.5. *Let $X = \text{Spec}(A)$ be an affine \mathbf{C} -scheme. Let M be a quasicoherent sheaf on $X^2 \times \mathbf{A}^1$. If the two copies of the A -actions are identical on M/tM , then M extends uniquely to a $N_{X^2} \Delta$ module, where Δ denotes the diagonal.*

Proof. Easy exercise by unraveling the Rees construction. \square

1.8. **Free objects in $\mathcal{HC}_{\hbar}^{\text{fg}}$.** We will mostly need κ_{\hbar} for some more special Harish-Chandra bimodules. Let V be a finite-dimensional representation of \check{G} . Define an associated \hbar -Harish-Chandra bimodule

$$\text{Fr}(V) := U_{\hbar} \otimes_{\mathbf{C}} V$$

where the \check{G} -action and the U_{\hbar}^2 -action are given by

$$g : u \otimes v \mapsto \text{Ad}(g)(u) \otimes gv, \quad x \otimes y : u \otimes v \mapsto xuy \otimes v + \hbar \cdot xu \otimes yv,$$

where yv is given by the action of \mathfrak{g} on V induced by the G -action. This functor $\text{Fr} : \text{Rep}(\check{G}) \rightarrow \mathcal{HC}_{\hbar}^{\text{fg}}$ is the left adjoint to the forgetful functor and can thus be called “induction.” Denote

$$\phi(V) := \kappa_{\hbar}(\text{Fr}(V)).$$

The objects in the essential image of the functor Fr are called free, and the full subcategory generated by the essential image is denoted $\mathcal{HC}_{\hbar}^{\text{fr}} \subset \mathcal{HC}_{\hbar}^{\text{fg}}$.

1.9. **Equivariant cohomology of Gr_G .** Equivariant cohomology is the “fiber functor” in our context. We start by computing some of the easiest ones.

Lemma 1.6. *We have*

$$R\Gamma_{L+G}(*, \mathbf{C}) \simeq R\Gamma_G(*, \mathbf{C}) \simeq \text{Sym}^{\square}(\mathfrak{g}^*)^G \simeq \text{Sym}^{\square}(\mathfrak{t}^*)^W,$$

where by \square we mean that we form symmetric power as differential graded algebras and elements in \mathfrak{g}^* and \mathfrak{t}^* are in degree 2. Moreover,

$$R\Gamma_{L+G \rtimes \mathbf{G}_m}(*, \mathbf{C}) \simeq R\Gamma_{L+G}(*, \mathbf{C}) \otimes^{\mathbf{L}} R\Gamma_{\mathbf{G}_m}(*, \mathbf{C}) \simeq \text{Sym}^{\square}(\mathfrak{t}^*)^W[\hbar],$$

where \hbar sits in degree 2.

Proof sketch. The first isomorphism follows from the fact that the kernel of $L^+G \rightarrow G$ (evaluation at $t = 0$) is pro-unipotent and has trivial cohomology. It is a standard fact that $R\Gamma(BG, \mathbf{C}) \simeq R\Gamma(BT, \mathbf{C})^W$. The rest follows from an easy computation of the group cohomology of a torus. The second statement follows the same argument since the loop rotation \mathbf{G}_m -action only happens on the pro-unipotent part. \square

Induced by $L^+G \backslash LG \rightarrow *$ and by $LG/L^+G \rightarrow *$, we have two natural projection maps

$$\mathrm{pr}_1, \mathrm{pr}_2 : L^+G \backslash LG / L^+G \rightrightarrows BL^+G.$$

This induces two projection maps on equivariant cohomology

$$\mathrm{pr}_1^*, \mathrm{pr}_2^* : \mathrm{Sym}^\square(\mathfrak{t}^*)^W \hookrightarrow \mathrm{Sym}^\square(\mathfrak{t}^*)^W[\hbar] \simeq R\Gamma_{L^+G \rtimes \mathbf{G}_m}(*) \rightrightarrows R\Gamma_{\mathbf{G}_m}(L^+G \backslash LG / L^+G) \simeq R\Gamma_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}_G).$$

The map $L^+G \backslash \mathrm{Gr}_G \rightarrow *$ induces

$$\mathrm{pr}^* : \mathbf{C}[\hbar] \simeq R\Gamma_{\mathbf{G}_m}(*) \rightarrow \mathbf{R}\Gamma_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}_G).$$

The above construction also makes clear that there is an action of $H^*(BL^+G)^2 \times H^*(B\mathbf{G}_m) \simeq \mathbf{C}[(\mathfrak{t}/W)^2 \times \mathbf{A}^1]$ on $H_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}_G)$.

Theorem 1.7. *There is a canonical isomorphism*

$$H_{L^+G \rtimes \mathbf{G}_m}^*(\mathrm{Gr}_G) \simeq \Gamma(N_{(\mathfrak{t}^*/W)^2} \Delta, \mathcal{O}),$$

where Δ denotes the diagonal and both sides are viewed as graded algebras where in the Rees construction on the right hand side we let elements in \mathfrak{t}^* and \hbar have degree 2. Under this isomorphism, pr^* , pr_1^* , and pr_2^* correspond to the composition of

$$N_{(\mathfrak{t}^*/W)^2} \Delta \rightarrow \mathbf{A}^1 \times (\mathfrak{t}^*/W)^2 \simeq \mathbf{A}^1 \times (\mathfrak{t}/W)^2$$

with projections onto the three factors, and this requirement pins down the isomorphism uniquely.

Remark 1.8 (Non-(simply connected) G). This is the first place where we use the assumption that G is simply connected. When G is no longer assumed to be simply connected, there is a canonical isomorphism

$$H_{L^+G \rtimes \mathbf{G}_m}^*(\mathrm{Gr}_G) \simeq \bigoplus_{\pi_1(G)} \Gamma(N_{(\mathfrak{t}^*/W)^2} \Delta, \mathcal{O}).$$

It is possible to adjust later results and proofs to accommodate for general G , but we will not pursue that in this note.

Remark 1.9 (Equivariant cohomology functor $\mathcal{H}_{L^+G \rtimes \mathbf{G}_m}^*$). Since the cohomology of any complex on a topological space carries a graded action by the cohomology of the constant sheaf, Theorem 1.7 implies that there is an equivariant cohomology functor

$$\mathcal{H}_{L^+G \rtimes \mathbf{G}_m}^* : \mathcal{D}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}_G) \rightarrow \mathrm{Coh}_{\mathbf{G}_m}(N_{(\mathfrak{t}^*/W)^2} \Delta),$$

where the \mathbf{G}_m -action is given by multiplication by t^2 on elements of \mathfrak{t}^* and on \hbar .

Theorem 1.10. *The Satake functor $\mathbf{S} : \mathrm{Rep}(\check{G}) \rightarrow \mathrm{Perv}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}_G)$ extends to a fully faithful functor*

$$\mathbf{S}_{\hbar} : \mathcal{H}_{\hbar}^{\mathrm{fr}} \rightarrow \mathcal{D}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}_G)$$

such that

$$\kappa_{\hbar} \simeq \mathcal{H}_{L^+G \rtimes \mathbf{G}_m}^* \circ \mathbf{S}_{\hbar}.$$

Once this isomorphism is specified, such an extension \mathbf{S}_{\hbar} is unique.

Remark 1.11. Since the functor $\mathrm{Fr} : \mathrm{Rep}(\check{G}) \rightarrow \mathcal{H}_{\hbar}^{\mathrm{fr}}$ is by definition essentially surjective, the essential image of \mathbf{S}_{\hbar} is nothing but $\mathrm{Perv}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}_G)$. Here is a potential source of confusion: it seems on the surface that since \mathbf{S} is an equivalence between (ordinary) categories, for \mathbf{S}_{\hbar} to be fully faithful, Fr must be an equivalence, which is absurd.

To resolve the confusion (and to clarify the statement of the theorem), it is very important to view $\mathcal{H}_{\hbar}^{\mathrm{fr}}$ as the full dg-subcategory (or ∞ -category) of $\mathcal{D}_{\mathrm{perf}, \check{G}}(U_{\hbar}^{\square})$ where U_{\hbar}^{\square} is a dga with zero differentials and with elements of \mathfrak{g} and \hbar in degree 2. Hence $\mathcal{H}_{\hbar}^{\mathrm{fr}}$ here is really a dg-category and the theorem states that \mathbf{S}_{\hbar} is a fully faithful functor between dg-categories. In particular, the adjoint functor $\mathrm{Fr} : \mathrm{Rep}(\check{G}) \rightarrow \mathcal{H}_{\hbar}^{\mathrm{fr}}$ induces an equivalence onto the underlying degree-0 ordinary category of the dg category. The failure of Fr to be an equivalence of ordinary categories on the nose is due to the fact that extra morphisms in $\mathcal{H}_{\hbar}^{\mathrm{fr}}$ are of higher degrees. This phenomenon is sometimes known as “shearing.”

With all notations introduced, we may state the main theorem.

Theorem 1.12 (Derived geometric Satake). *There exist canonical equivalence of monoidal stable ∞ -categories*

$$\mathcal{D}_{\text{perf}, \check{G}}(U_h^\square) \simeq \mathcal{D}_{L+G \rtimes \mathbf{G}_m}(\text{Gr}_G), \quad \mathcal{D}_{\text{perf}, \check{G}}(\text{Sym}^\square(\check{\mathfrak{g}})) \simeq \mathcal{D}_{L+G}(\text{Gr}_G).$$

Remark 1.13 (Derived Satake is not t -exact). The equivalences above are not t -exact with respect to the standard t -structure on the left and the perverse t -structure on the right. As we already saw before, it should restrict to an isomorphism between

$$\mathcal{H}(\mathcal{C}_h^{\text{fr}} \simeq \text{Perv}_{L+G \rtimes \mathbf{G}_m}(\text{Gr}_G))$$

as full dg-subcategories, and the left-hand side is very far from being discrete since U_h^\square is not discrete.

Remark 1.14 (Koszul duality). By Koszul duality, derived Satake implies that there is an equivalence

$$\mathcal{D}_{L+G}(\text{Gr}_G) \simeq \mathcal{D}_{\text{perf}}((0 \times_{\check{\mathfrak{g}}}^{\mathbf{L}} 0)/\check{G}).$$

This formulation is probably closer to the spirit of geometric Langlands.

1.10. Quasiclassical limit. Let's say a few words on a form of Theorem 1.10 for $\mathcal{D}_{L+G}(\text{Gr}_G)$ via some limiting procedure. Recall that the fiber of $N_X Z$ over $0 \in \mathbf{A}^1$ is the normal cone to Z in X . In particular, the fiber of $N_{(\check{\mathfrak{t}}^*//W)^2} \Delta$ over 0 is the total space of the tangent bundle $\mathbf{T}(\check{\mathfrak{t}}^*//W)$. Hence, Theorem 1.7 formally implies the following corollary.

Corollary 1.15. *There is a canonical isomorphism*

$$H_{L+G}^*(\text{Gr}_G) \simeq \mathbf{C}[\mathbf{T}(\check{\mathfrak{t}}^*//W)].$$

Consider the composition

$$\text{pr} : (\check{\mathfrak{g}}^*)^{\text{reg}} \hookrightarrow \check{\mathfrak{g}}^* \rightarrow \check{\mathfrak{g}}^*//\check{G} \simeq \check{\mathfrak{t}}^*//W.$$

Denote

$$\mathfrak{z} := \text{pr}^*(T^*(\check{\mathfrak{t}}^*//W))$$

the pullback of the cotangent bundle. Unraveling the definition, \mathfrak{z} is a subsheaf of commutative Lie algebras $\mathfrak{z} \subset \check{\mathfrak{g}} \otimes_{\mathbf{C}} \mathcal{O}_{\check{\mathfrak{g}}^*}$ whose fiber at $\xi \in (\check{\mathfrak{g}}^*)^{\text{reg}}$ is the stabilizer of ξ : the fiber is dual to the cokernel of

$$\check{\mathfrak{g}} \rightarrow \check{\mathfrak{g}}^*, \quad x \mapsto \text{coad}(x)(\xi).$$

Now we define some variants of Kostant functors

$$\kappa : \text{Coh}_{\check{G} \times \mathbf{G}_m}(\check{\mathfrak{g}}^*) \rightarrow \text{Coh}_{\mathbf{G}_m}(\mathbf{T}(\check{\mathfrak{t}}//W)), \quad \bar{\kappa} : \text{Coh}_{\check{G}}(\check{\mathfrak{g}}^*) \rightarrow \text{Coh}(\mathbf{T}(\check{\mathfrak{t}}//W)).$$

We construct $\bar{\kappa}$ first. For $\mathcal{M} \in \text{Coh}_{\check{G}}(\check{\mathfrak{g}}^*)$, let \mathcal{M} also denote the underlying coherent sheaf on $\check{\mathfrak{g}}^*$. The \check{G} -equivariance structure endows $\mathcal{M}|_{(\check{\mathfrak{g}}^*)^{\text{reg}}}$ with a \mathfrak{z} -action. Let $\Sigma \subset \check{\mathfrak{g}}^*$ be the image of the Kostant slice $e + \mathfrak{z}_{\mathfrak{g}}(f) \subset \check{\mathfrak{g}}$ under a \check{G} -invariant isomorphism $\check{\mathfrak{g}} \simeq \check{\mathfrak{g}}^*$. There is a composition

$$j : \check{\mathfrak{t}}^*//W \simeq \Sigma \hookrightarrow (\check{\mathfrak{g}}^*)^{\text{reg}},$$

which clearly satisfies that $\text{pr} \circ j = \text{id}$. Thus $j^*\mathcal{M}|_{(\check{\mathfrak{g}}^*)^{\text{reg}}}$ carries an action of $T^*(\check{\mathfrak{t}}^*//W)$, giving a coherent sheaf on $\mathbf{T}^*(\check{\mathfrak{t}}^*//W)$. In this way we constructed $\bar{\kappa}(\mathcal{M})$.

The statement with \mathbf{G}_m -equivariant sheaves follows by the same proof, except for that we have to be careful with the \mathbf{G}_m -action on $\check{\mathfrak{g}}^*$. As discussed previously, the Lie algebra $\check{\mathfrak{g}}$ splits into a direct sum of $\check{\mathfrak{g}}(i)$ according to the adjoint action by h . Consider the \mathbf{G}_m -action on $\check{\mathfrak{g}}$ that is given by

$$t \cdot v := t^{2+i}v, \quad v \in \check{\mathfrak{g}}(i).$$

This action fixes $f \in \check{\mathfrak{g}}$ in the \mathfrak{sl}_2 -triple. This action is compatible with the grading endowed on $v \in \check{\mathfrak{g}}(i)$ in U_h . Induced by this action there is a natural \mathbf{G}_m -action on $\check{\mathfrak{g}}^*$, where the action on $\check{\mathfrak{g}}(i)^*$ is multiplication by t^{-2-i} . Note that under a G -equivariant isomorphism $\check{\mathfrak{g}} \simeq \check{\mathfrak{g}}^*$, we must have

$$\check{\mathfrak{g}}(i) \leftrightarrow \check{\mathfrak{g}}(-i)^*.$$

Hence the image e in $\check{\mathfrak{g}}^*$ lies in $\check{\mathfrak{g}}(-2)^*$ and is fixed by the \mathbf{G}_m -action. Hence the image of the Kostant slice $\Sigma \subset \check{\mathfrak{g}}^*$ is fixed by the \mathbf{G}_m -action. This ensures the construct above goes through for \mathbf{G}_m -equivariant sheaves.

Define $\text{Coh}_{\check{G} \times \mathbf{G}_m}^{\text{fr}}(\check{\mathfrak{g}}^*) \subset \text{Coh}_{\check{G} \times \mathbf{G}_m}(\check{\mathfrak{g}}^*)$ the full subcategory spanned by objects of the form $V \otimes_{\mathbf{C}} \mathcal{O}_{\check{\mathfrak{g}}^*}$ for $V \in \text{Rep}(\check{G} \times \mathbf{G}_m)$. These are coherent sheaves pulled back from $B(\check{G} \times \mathbf{G}_m)$. The functor of L^+G -equivariant cohomology gives a functor

$$\mathcal{H}_{L^+G}^* : \mathcal{D}_{L^+G}(\text{Gr}_G) \rightarrow \text{Coh}_{\mathbf{G}_m}(\mathbf{T}(\check{\mathfrak{t}}^//W)).$$

Theorem 1.16. *The Satake functor $\mathbf{S} : \text{Rep}(\check{G}) \rightarrow \text{Perv}_{L^+G}(\text{Gr}_G)$ extends to a fully faithful embedding*

$$\mathbf{S}_{\text{qc}} : \text{Coh}_{\check{G} \times \mathbf{G}_m}^{\text{gr}}(\check{\mathfrak{g}}^*) \rightarrow \mathcal{D}_{L^+G}(\text{Gr}_G)$$

such that there is an isomorphism

$$\kappa \simeq \mathcal{H}_{L^+G}^* \circ \mathbf{S}_{\text{qc}}.$$

2. TOPOLOGICAL FILTRATIONS

By ‘‘topological’’ we mean data extracted from the automorphic side, i.e., by considering bounded constructible sheaves on Gr_G . By ‘‘algebraic’’ we will mean data extracted from the spectral side, i.e., by considering \hbar -Harish-Chandra bimodules and variants. From now on, the exposition will be less detailed than before. We will not prove Theorem 1.7 apart from remarking that some localization theorem for torus-equivariant cohomology is a major ingredient.

A few remarks on equivariant cohomology are in order. If U is a unipotent algebraic group acting on a variety X , then

$$R\Gamma_U(X, \mathcal{F}) \simeq R\Gamma(X, \mathcal{F})$$

is a natural isomorphism for a U -equivariant sheaf \mathcal{F} . Consequently, there is a natural isomorphism

$$R\Gamma_{L^+G \rtimes \mathbf{G}_m}(\text{Gr}_G, \mathcal{F}) \simeq R\Gamma_{G \times \mathbf{G}_m}(\text{Gr}_G, \mathcal{F})$$

for a $L^+G \rtimes \mathbf{G}_m$ -equivariant perverse sheaf \mathcal{F} . Smooth base change then guarantees that

$$R\Gamma_{T \times \mathbf{G}_m}(\text{Gr}_G, \mathcal{F}) \simeq R\Gamma_{G \times \mathbf{G}_m}(\text{Gr}_G, \mathcal{F}) \otimes_{\mathbf{C}[\mathfrak{t}/W]} \mathbf{C}[\mathfrak{t}].$$

This implies that there is a natural action of $\mathbf{C}[\mathfrak{t} \times (\mathfrak{t}/W) \times \mathbf{A}^1]$ on $H_{T \times \mathbf{G}_m}^*(\text{Gr}_G, \mathcal{F})$.

Let’s fix some notations. Denote

$$\pi : \mathfrak{t} \rightarrow \mathfrak{t}/W.$$

Let $\mu \in \mathbf{X}_*(T)$. Recall that the LN_- -orbits on Gr_G is parametrized by $\mathbf{X}_*(T)$ and the semi-infinite orbits are defined as

$$T_\mu := LN_- \cdot t^\mu L^+G \subset \text{Gr}_G.$$

Note that we are working with LN_- -orbits instead of LN -orbits. To understand the relationship, let $w \in W$ be the element sending B to B_- . Pick a lift $\dot{w} \in G(\mathbf{C})$ of w , then

$$\dot{w} \cdot S_\mu = \dot{w} \cdot LN \cdot t^\mu L^+G = LN_- \cdot \dot{w} \cdot t^\mu L^+G = LN_- \cdot t^{w(\mu)} L^+G.$$

Hence

$$w(S_\mu) = T_{w(\mu)}.$$

In particular under w the Bruhat order is reversed and hence the closure of T_μ is

$$\overline{T_\mu} = T_{\geq \mu} = \bigcup_{\lambda \geq \mu} T_\lambda$$

on the level of \mathbf{C} -points. Denote

$$j_\lambda : t^\lambda \hookrightarrow T_\lambda, \quad \iota_\lambda : T_\lambda \hookrightarrow \text{Gr}_G.$$

2.1. **The Mirkovic-Vilonen filtration on $H_{T \times \mathbf{G}_m}^*(\mathbf{Gr}_G, \mathcal{F})$.** For $\lambda \in \mathbf{X}_*(T)$, there is a natural map

$$r_\lambda : H_{T \geq \lambda, T \times \mathbf{G}_m}^*(\mathbf{Gr}_G, \mathcal{F}) \rightarrow H_{T \times \mathbf{G}_m}^*(\mathbf{Gr}_G, \mathcal{F})$$

where the left hand side is cohomology with support (or relative cohomology relative to the complement in the language of singular cohomology). The images of r_λ defines a $\mathbf{X}_*(T)$ -filtration on $H_{T \times \mathbf{G}_m}^*(\mathbf{Gr}_G, \mathcal{F})$. By hyperbolic localization¹ the associated graded of this filtration is

$$\bigoplus_{\lambda \in \mathbf{X}_*(T)} H_{T_\lambda, T \times \mathbf{G}_m}^*(\mathbf{Gr}_G, \mathcal{F}) \simeq \bigoplus_{\lambda \in \mathbf{X}_*(T)} H_{T \times \mathbf{G}_m}^*(t^\lambda, \mathbf{C}) \otimes_{\mathbf{C}} j_{\lambda!}^* \mathcal{F}.$$

Here the isomorphism is also an isomorphism of quasicoherent sheaves on $\mathfrak{t} \times (\mathfrak{t}/W) \times \mathbf{A}^1$. Note that t^μ is fixed by the \mathbf{G}_m action since \mathbf{G}_m scales t while the scalar is absorbed by right quotienting L^+G . It is again proved by Mirkovic-Vilonen (as a property of their fiber functor) that when $\mathcal{F} \simeq \mathbf{S}(V)$ for some $V \in \text{Rep}(\check{G})$,

$$j_{\lambda!}^* \mathcal{F} \simeq V_\lambda,$$

where $V_\lambda \subset V$ is the weight- λ subspace.

Remark 2.1 (Constant term functor). As discussed in Jacob's talk, there is a natural isomorphism

$$H_{T_\lambda}^*(\mathcal{F}) \simeq H_c^*(S_\lambda, \mathcal{F}|_{S_\lambda}) \simeq j_{\lambda!}^* \mathcal{F} \simeq V_\lambda.$$

It remains to say something about $H_{T \times \mathbf{G}_m}^*(t^\mu, \mathbf{C})$ as a sheaf on $\mathfrak{t} \times (\mathfrak{t}/W) \times \mathbf{A}^1$. Define $\Gamma_\lambda \subset \mathfrak{t} \times \mathfrak{t} \times \mathbf{A}^1$ by

$$\Gamma_\lambda := \{(x_1, x_2, a) \in \mathfrak{t} \times \mathfrak{t} \times \mathbf{A}^1 : x_2 = x_1 + a\lambda\},$$

where λ is a cocharacter of T , i.e., a point on \mathfrak{t} .

Lemma 2.2. *We have*

$$H_{T \times \mathbf{G}_m}^*(t^\mu) \simeq (\text{id}, \pi, \text{id})_* \mathcal{O}_{\Gamma_\lambda}.$$

Remark 2.3. Let's do a sanity check. When restricted to $\mathfrak{t} \times \{0\} \times \mathbf{A}^1$, the right hand side is clearly the structure sheaf. This agrees with the left hand side since $t^\mu \in \text{Gr}_G$ is nothing but a point so its $T \times \mathbf{G}_m$ -equivariant cohomology is not supposed to say anything interesting.

Remark 2.4. Restricting to the first and the third factor, we learn that $H_{T \times \mathbf{G}_m}^*(\mathcal{F})$ admits a filtration with graded pieces free as $\mathbf{C}[\mathfrak{t} \times \mathbf{A}^1]$ -modules and hence $H_{T \times \mathbf{G}_m}^*(\mathcal{F})$ is flat as a $\mathbf{C}[\mathfrak{t} \times \mathbf{A}^1]$ -module. Hence $H_{L^+G \rtimes \mathbf{G}_m}^*(\mathcal{F})$ is flat as $\mathbf{C}[\mathfrak{t}/W \times \mathbf{A}^1]$ by descent of flatness along integral morphisms.

2.2. **Restriction to Levi subgroups.** The Mirkovic-Vilonen filtration varies compatibly when we restrict to Levi subgroups. Let $T \subset M \subset G$ be a Levi subgroup and let $P \subset G$ be the parabolic subgroup generated by M and B with unipotent radical N_M . Then the obvious inclusion $N_M(T) \subset N_G(T)$ induces an inclusion $W_M \subset W$ where W is the Weyl group of M . Since T is a maximal torus in M and $B \cap M$ is a Borel in M , on the dual side $(\check{B} \cap \check{M}, \check{T})$ is a Borel pair for M . They determine a set of dominant weights

$$\mathbf{X}^*(\check{T})_M^+ \supset \mathbf{X}^*(\check{T})^+.$$

Let

$$\pi_L : \mathfrak{t}/W_M \rightarrow \mathfrak{t}/W$$

denote the natural projection.

Lemma 2.5. *For $V \in \text{Rep}(\check{G})$, there is a natural filtration on $(\pi_L, \text{id}, \text{id})^* H_{L^+G \rtimes \mathbf{G}_m}^*(\mathbf{Gr}_G, \mathbf{S}(V))$ that satisfies the following natural isomorphism*

$$\Xi_M^{\text{top}} : \bigoplus_{\lambda \in \mathbf{X}^*(Z_M)} H_{M T_\lambda, M \times \mathbf{G}_m}^*(\mathbf{Gr}_G, \mathbf{S}(V)) \simeq (\text{id}, \pi_L, \text{id})_* H_{L^+M \rtimes \mathbf{G}_m}^*(\mathbf{Gr}_M, \mathbf{S}_M(V|_{\check{M}})).$$

¹Here we need a version of hyperbolic localization for equivariant sheaves. More precisely we mean a version of Proposition 24 in Jacob's notes for G -equivariant sheaves \mathcal{F} where the G -action commutes with the \mathbf{G}_m -action. The proof there carries to here without changes.

Remark 2.6 (Transitivity for a pair of Levi subgroups). The above isomorphism is compatible with further restriction of Levi subgroups. We only sketch the idea here. If $T \subset M' \subset M$ is another Levi, the pullback of the MV filtration on $H_{M \rtimes \mathbf{G}_m}(\mathrm{Gr}_G, \mathbf{S}(V))$ gives a filtration on $H_{M' \rtimes \mathbf{G}_m}(\mathrm{Gr}_G, \mathbf{S}(V))$. The MV filtration on $H_{M' \rtimes \mathbf{G}_m}(\mathrm{Gr}_G, \mathbf{S}(V))$ refines this pullback filtration and induces a filtration on the pullback of the associated graded of $H_{M \rtimes \mathbf{G}_m}(\mathrm{Gr}_G, \mathbf{S}(V))$ which comes from the MV filtration on $H_{M' \rtimes \mathbf{G}_m}(\mathrm{Gr}_M, \mathbf{S}(V|_{\check{M}}))$. One may show that the isomorphisms Ξ^{top} are compatible with the identification outlined above. We won't spell out what "compatible" means, but in this context, I suspect there is only one reasonable interpretation.

2.3. Monoidal structures. We will not say a great deal about the monoidal structures due to the scope of this note and the incompetence of the author, but let's sketch the main idea and make a few remarks.

As Longke explained in his talk, there is a very general monoidal structure on $\mathcal{D}(H \backslash G / H)$ for $H \subset G$ a sub-(algebraic group), where associativity follows from proper base change and the projection formula. For any bounded constructible sheaf \mathcal{F} on $H \backslash G / H$, pushing forward to $H \backslash *$ and $*/H$ gives $H^*(H \backslash G / H, \mathcal{F})$ a structure as of $H^*(BH)$ -bimodules. The functor sending \mathcal{F} to this bimodule is monoidal. As for the convolution product, recall that one may descend $\mathcal{F} \boxtimes \mathcal{G}$ to $\mathcal{F} \boxtimes^H \mathcal{G}$ on

$$H \backslash (G \times^H G) / H$$

and we may define

$$\mathcal{F} * \mathcal{G} := m_!(\mathcal{F} \boxtimes^H \mathcal{G})$$

where m is induced by multiplication on G .

Remark 2.7 (Comparison with the fiber functor in Longke's notes). While this functor

$$\mathcal{D}(H \backslash G / H) \rightarrow H^*(BG)\text{-BiMod}$$

is monoidal, just remembering the $H^*(BG)$ -action coming from projecting to $*/H$ will not automatically give a monoidal functor. More precisely, as Longke explained, the pull-push along the correspondence

$$H \backslash G / H \leftarrow G / H \rightarrow *$$

is *not* in general monoidal. One need either to use the fusion product structure or argue as in [Zhu16, §5.2] that the left and right $H^*(BG)$ -actions on the equivariant cohomology agree.

We are interested in making the functor

$$\mathrm{Rep}(\check{G}) \xrightarrow{\mathbf{S}} \mathrm{Perv}_{L^+G \rtimes \mathbf{G}_m}(\mathrm{Gr}_G) \xrightarrow{H^*_{L^+G \rtimes \mathbf{G}_m}} \mathrm{QCoh}(t//W \times t//W \times \mathbf{A}^1)$$

monoidal. We already know that \mathbf{S} is monoidal:

$$\mathbf{S}(V_1 \otimes V_2) \simeq \mathbf{S}(V_1) * \mathbf{S}(V_2).$$

To make the hypercohomology functor monoidal, one just needs to define the correct monoidal structure on $\mathrm{Coh}(t//W \times t//W \times \mathbf{A}^1)$ and then the proof is easy. Let's explain this monoidal structure a bit more.

Construction 2.8 (A monoidal structure on $\mathrm{QCoh}(t//W \times t//W \times \mathbf{A}^1)$). Recall that for a sheaf \mathcal{F} on $(L^+G \rtimes \mathbf{G}_m) \backslash LG / L^+G \simeq (L^+G \rtimes \mathbf{G}_m) \backslash (LG \rtimes \mathbf{G}_m) / (L^+G \rtimes \mathbf{G}_m) \simeq L^+G \backslash (LG \rtimes \mathbf{G}_m) / (L^+G \rtimes \mathbf{G}_m)$, the cohomology $H^*(\mathcal{F})$ naturally acquires the structure of a $H^*(BG) \times H^*(B\mathbf{G}_m) \simeq \mathbf{C}[t//W \times \mathbf{A}^1]$ -bimodule structure. One may transport the bi-module structure to a $\mathbf{C}[(t//W)^2 \times \mathbf{A}^2]$. Unravelling the definition, one sees that the two factors of $\mathbf{C}[\mathbf{A}^1]$ -actions differ by $t \mapsto -t, \mathbf{A}^1 \rightarrow \mathbf{A}^1$. Forgetting the action of the second factor, we recover the $\mathbf{C}[(t//W)^2 \times \mathbf{A}^1]$ -module structure on $H^*(\mathcal{F})$ introduced before. Twisting the second \mathbf{A}^1 -action by this involution we obtain a new $\mathbf{C}[t//W \times \mathbf{A}^1]$ -bimodule structure, and this is the correct bimodule structure to take tensor product, which takes two $\mathbf{C}[t//W \times \mathbf{A}^1]$ -bimodule coming from one $\mathbf{C}[(t//W)^2 \times \mathbf{A}^1]$ and produces their tensor product, another $\mathbf{C}[t//W \times \mathbf{A}^1]$ -bimodule coming from a $\mathbf{C}[(t//W)^2 \times \mathbf{A}^1]$ -module.

More concretely, let X be qcqs over \mathbf{C} . For two quasicohherent sheaves \mathcal{M} and \mathcal{N} on $X \times X \times \mathbf{A}^1$, consider the maps

$$p_{12}, p_{23}, p_{13} : X \times X \times X \times \mathbf{A}^1 \rightarrow X \times X \times \mathbf{A}^1.$$

One may form convolution

$$\mathcal{M} * \mathcal{N} := p_{13,*}(p_{12}^* \mathcal{M} \otimes p_{23}^* \mathcal{N}).$$

Under this convolution product, we have

$$\mathcal{O}_{\Gamma_\lambda} * \mathcal{O}_{\Gamma_\mu} \simeq \mathcal{O}_{\Gamma_{\lambda+\mu}}.$$

Lemma 2.9. *We have*

$$H_{L+G \times \mathbf{G}_m}^*(\mathcal{F} * \mathcal{G}) \simeq H_{L+G \times \mathbf{G}_m}^*(\mathcal{F}) * H_{L+G \times \mathbf{G}_m}^*(\mathcal{G})$$

as quasicoherent sheaves on $\mathfrak{t} // W \times \mathfrak{t} // W \times \mathbf{A}^1$.

Remark 2.10 (Compatibility on the generic fiber). Finally we are interested in some compatibility between the above isomorphism on the generic fiber, where the grading on $H_{T \times \mathbf{G}_m}^*$ (say, by Atiyah-Bott's localization theorem for torus equivariant cohomology [GKM98, 6.2]). Namely, base changing to the generic fiber on the first and third factors of $\mathfrak{t} // W \times \mathfrak{t} // W \times \mathbf{A}^1$, we get a natural isomorphism

$$\Xi_{V,\eta}^{\text{top}} : H_{L+G \times \mathbf{G}_m}(\text{Gr}_G, \mathbf{S}(V)) \otimes_{\mathbf{C}[\mathfrak{t} // W \times \mathbf{A}^1]} \mathbf{C}(\mathfrak{t} \times \mathbf{A}^1) \simeq \bigoplus_{\lambda \in \mathbf{X}_*(T)} (\mathcal{O}_{\Gamma_\lambda} \otimes_{\mathbf{C}[\mathfrak{t} \times \mathbf{A}^1]} \mathbf{C}(\mathfrak{t} \times \mathbf{A}^1)) \otimes_{\mathbf{C}} V_\lambda.$$

One may take convolution products on both sides of $\Xi_{V_1,\eta}^{\text{top}}$ and $\Xi_{V_2,\eta}^{\text{top}}$ and wonder whether this will give $\Xi_{V_1 \otimes V_2, \eta}^{\text{top}}$. One may make this question precise in a unique way and (nontrivially!) verify it. We won't say more about the details.

3. ALGEBRAIC FILTRATIONS

A similar story can be developed for \hbar -Harish-Chandra modules, i.e., on the algebraic/spectral side. Recall that by some explicit procedure, we have defined Kostant functors

$$\kappa_{\hbar} : \mathcal{H}\mathcal{C}_{\hbar}^{\text{fg}} \rightarrow \text{QCoh}_{\mathbf{G}_m}((\mathfrak{t} // W)^2 \times \mathbf{A}^1), \quad \kappa : \text{Coh}_{\check{G} \times \mathbf{G}_m}(\check{\mathfrak{g}}^*) \rightarrow \text{Coh}_{\mathbf{G}_m}(\mathbf{T}(\check{\mathfrak{t}} // W)).$$

The following lemma is essentially proved by Kostant, and we will take it on faith.

Lemma 3.1. *The functors κ and κ_{\hbar} are exact. Moreover, the functors $\kappa|_{\text{Coh}_{\check{G} \times \mathbf{G}_m}^{\text{fg}}(\check{\mathfrak{g}}^*)}$ and $\kappa_{\hbar}|_{\mathcal{H}\mathcal{C}_{\hbar}^{\text{fg}}}$ are fully faithful.*

3.1. Desymmetrized Kostant functor \varkappa_{\hbar} and a canonical filtration. Recall that we defined $\phi(V) := \kappa_{\hbar}(\text{Fr}(V))$ for $V \in \text{Rep}(\check{G})$. We are interested in describing $(\pi, \text{id}, \text{id})^* \phi(V) \in \text{Coh}(\mathfrak{t} \times \mathfrak{t} // W \times \mathbf{A}^1)$.

Construction 3.2 (Universal Verma module). We need to first introduce the universal Verma module. One may view $\mathbf{C}[\hbar][\check{\mathfrak{t}}]$ to be a $U_{\hbar}(\check{\mathfrak{t}})$ -module where $t \in \check{\mathfrak{t}}$ acts by multiplication by $t - \hbar\rho(t)$ where ρ as usual is the halfsum of positive roots of $\check{\mathfrak{g}}$. One may extend this action to a $U_{\hbar}(\check{\mathfrak{b}})$ -action via projection $\check{\mathfrak{b}} \rightarrow \check{\mathfrak{t}}$. We call this module $\mathbf{C}[\hbar][\check{\mathfrak{t}}][-\rho]$ as an $U_{\hbar}(\check{\mathfrak{b}})$ -module. Define the *universal Verma module* to be

$$\mathcal{M}_{\hbar}(-\rho) := U_{\hbar} \otimes_{U_{\hbar}(\check{\mathfrak{b}})} \mathbf{C}[\hbar][\check{\mathfrak{t}}][-\rho].$$

Construction 3.3 (Desymmetrized Kostant functor \varkappa_{\hbar}). Let M be a \hbar -Harish-Chandra bimodule, define

$$\varkappa_{\hbar}(M) := \mathcal{M}_{\hbar}(-\rho) \otimes_{U_{\hbar}}^{\mathbf{L}} M \otimes_{1 \otimes U_{\hbar}(\check{\mathfrak{n}}_-), -\psi}^{\mathbf{L}} \mathbf{C}[\hbar]$$

where $\psi : U_{\hbar}(\check{\mathfrak{n}}_-) \rightarrow \mathbf{C}[\hbar]$ is defined by sending a set of simple root generators to 1. One may see that $U_{\hbar}(\check{\mathfrak{t}})$ acts on the first factor on $\varkappa_{\hbar}(M)$ and still $Z(U_{\hbar})$ acts on the second factor. This endows $\varkappa_{\hbar}(M)$ a structure of an $\mathbf{C}[\mathfrak{t} \times \mathfrak{t} // W \times \mathbf{A}^1]$ -module.

For $V \in \text{Rep}(\check{G})$, let's denote $\varphi(V) := \varkappa_{\hbar}(\text{Fr}(V))$.

Lemma 3.4. *There is a canonical isomorphism $\varphi(V) \simeq (\pi, \text{id}, \text{id})^* \phi(V)$.*

Construction 3.5 (A canonical filtration on $\varphi(V)$). When working with $V \in \text{Rep}(\check{G})$ the relevant \hbar -Harish-Chandra bimodules are free and hence we don't need to worry about derived tensor products. Unravelling the definition we have

$$\varphi(V) \simeq \mathcal{M}_{\hbar}(-\rho) \otimes_{U_{\hbar}}^{\mathbf{L}} \text{Fr}(V) \otimes_{1 \otimes U_{\hbar}(\check{\mathfrak{n}}_-), -\psi}^{\mathbf{L}} \mathbf{C}[\hbar] \simeq (\mathcal{M}_{\hbar}(-\rho) \otimes_{\mathbf{C}} V) \otimes_{U_{\hbar}(\check{\mathfrak{n}}_-), \psi} \mathbf{C}[\hbar].$$

We have

$$\mathcal{M}_{\hbar}(-\rho) \otimes_{\mathbf{C}} V \simeq U_{\hbar} \otimes_{U_{\hbar}(\check{\mathfrak{n}}_-)} (\mathbf{C}[\hbar][\check{\mathfrak{t}}][-\rho] \otimes_{U_{\hbar}(\check{\mathfrak{b}})} V).$$

Note that

$$V_{\geq \lambda} := \bigoplus_{\mu \geq \lambda} V_{\mu}$$

is $\check{\mathfrak{b}}$ -stable. We may canonically filter V by $\{V_{\geq \lambda}\}_{\lambda \in \mathbf{X}^*(\check{\mathfrak{i}})}$ as $\check{\mathfrak{b}}$ -representations. This induces a filtration on $\mathcal{M}_{\check{h}}(-\rho) \otimes_{\mathbf{C}} V$ and hence on $\varphi(V)$. The associated graded of this filtration is

$$\bigoplus_{\lambda \in \mathbf{X}^*(\check{\mathfrak{i}})} (\mathcal{M}_{\check{h}}(\lambda - \rho) \otimes_{U_{\check{h}}(\check{\mathfrak{n}}_-), \psi} \mathbf{C}[\check{h}]) \otimes_{\mathbf{C}} V_{\lambda}.$$

Note that the grading is indexed by $\mathbf{X}^*(\check{\mathfrak{i}}) = \mathbf{X}_*(\mathfrak{t}) = \mathbf{X}_*(T)$, compatible with the grading on the automorphic/topological side. The $U_{\check{h}}(\mathfrak{t})$ - $Z(U_{\check{h}})$ -bimodule structure on $\mathcal{M}_{\check{h}}(\lambda - \rho) \otimes_{U_{\check{h}}(\check{\mathfrak{n}}_-), \psi} \mathbf{C}[\check{h}]$ makes sure that this is a grading by coherent sheaves on $\mathfrak{t} \times \mathfrak{t}/W \times \mathbf{A}^1$. Similar to the automorphic/topological side, we may identify the λ -piece of this coherent sheaf with $(\text{id}, \pi, \text{id})_* \mathcal{O}_{\Gamma_{\lambda}}$.

We thus proved that $\varphi(V)$ admits a canonical filtration with associated graded

$$\bigoplus_{\lambda \in \mathbf{X}^*(\check{\mathfrak{i}})} (\text{id}, \pi, \text{id})_* \mathcal{O}_{\Gamma_{\lambda}} \otimes_{\mathbf{C}} V_{\lambda}.$$

It is a general fact of representation theory that this grading admits a canonical splitting when base changed to the generic fiber on the first and third factors of $\mathfrak{t} \times \mathfrak{t}/W \times \mathbf{A}^1$. This is parallel to the generic splitting phenomenon for torus-equivariant cohomology.

3.2. Restriction to Levi subalgebras. Parallel to the automorphic/topological side, the filtration is compatible with restriction to Levi subalgebras in a precise sense. We only sketch the ideas here.

Let $\check{\mathfrak{i}} \subset \check{\mathfrak{m}} \subset \check{\mathfrak{g}}$ be a Levi subalgebra corresponding to a Levi subgroup M with Weyl group $W_M \subset W$ and recall that

$$\pi_M : \mathfrak{t}/W_M \rightarrow \mathfrak{t}/W$$

is the projection map. Then one may prove that for $V \in \text{Rep}(\check{G})$, the $\mathbf{C}[\check{\mathfrak{t}}/W_M \times \check{\mathfrak{t}}/W \times \mathbf{A}^1]$ -module $(\pi_M, \text{id}, \text{id})^* \phi(V)$ carries a canonical filtration such that its associated graded is equipped with a canonical isomorphism

$$\Xi_M^{\text{alg}} : \text{gr}^{\bullet}(\pi_M, \text{id}, \text{id})^* \phi(V) \simeq (\text{id}, \pi_M, \text{id})_* \phi_M(V|_{\check{M}}).$$

When $M = T$, this recovers the canonical filtration on $\varphi(V)$.

Remark 3.6 (Source of the filtrations). Recall that the source of the topological filtration is the stratification of the affine Grassmannians by its LN_{M-} -orbits. The source of the algebraic filtration comes down to the fact that for any parabolic \mathfrak{p} , V admits a \mathfrak{p} -stable filtration indexed by the weight lattice of the center of the Levi. When $\mathfrak{p} = \mathfrak{b}$ this recovers the \mathfrak{b} -stable filtration of V by $\mathbf{X}^*(\check{\mathfrak{i}})$.

One may again show that this isomorphism is compatible with passing to smaller Levi subalgebras. There is a unique way of making this precise and we won't say more about this.

3.3. Monoidal structures. Recall that we constructed a functor

$$\phi : \text{Rep}(\check{G}) \rightarrow \text{QCoh}((\mathfrak{t}/W)^2 \times \mathbf{A}^1)$$

and we have also explained a monoidal structure on $\text{QCoh}((\mathfrak{t}/W)^2 \times \mathbf{A}^1)$. One may verify that ϕ thus constructed is monoidal. One may also see, again parallel to the automorphic/topological side, the generic splitting property of the canonical filtration gives a canonical isomorphism

$$\Xi_{V, \eta}^{\text{alg}} : \phi(V) \otimes_{\mathbf{C}[\mathfrak{t}/W \times \mathbf{A}^1]} \mathbf{C}(\mathfrak{t} \times \mathbf{A}^1) \simeq \bigoplus_{\lambda \in \mathbf{X}^*(\check{\mathfrak{i}})} (\mathcal{O}_{\Gamma_{\lambda}} \otimes_{\mathbf{C}[\mathfrak{t}/W \times \mathbf{A}^1]} \mathbf{C}(\mathfrak{t} \times \mathbf{A}^1)) \otimes_{\mathbf{C}} V_{\lambda}.$$

One may again wonder whether tensoring two sides of $\Xi_{V_1, \eta}^{\text{alg}}$ and $\Xi_{V_2, \eta}^{\text{alg}}$ will give $\Xi_{V_1 \otimes V_2, \eta}^{\text{alg}}$. The answer is yes.

4. COMPUTATIONS FOR PGL_2

We do some explicit calculations for $G = \text{PGL}_2$. General groups G can be probed by the rank 1 cases via restriction to Levi subgroups. The dual of PGL_2 is SL_2 which is simply connected. It seems weird that BF uses PGL_2 as an example when they assume that G is simply connected, but this is basically harmless as $\text{SL}_2 \twoheadrightarrow \text{PGL}_2$ is a double cover, inducing a map

$$\text{Gr}_{\text{SL}_2} \rightarrow \text{Gr}_{\text{PGL}_2} = \text{Gr}_{\text{PGL}_2}^0 \sqcup \text{Gr}_{\text{PGL}_2}^1$$

that is an isomorphism onto the identity component $\mathrm{Gr}_{\mathrm{PGL}_2}^0$ of $\mathrm{Gr}_{\mathrm{PGL}_2}$. The $L^+\mathrm{SL}_2$ -action on $\mathrm{Gr}_{\mathrm{SL}_2} = \mathrm{Gr}_{\mathrm{PGL}_2}^0$ factors through the PGL_2 -action on $\mathrm{Gr}_{\mathrm{PGL}_2}$. The category $\mathrm{Rep}(\mathrm{SL}_2)$ is semisimple with simple objects V_n irreducible of dimension $n + 1$ and the category $\mathrm{Rep}(\mathrm{PGL}_2)$ is semisimple with simple objects V_{2n} on which the SL_2 -action descends to a PGL_2 -action. The Schubert cells Gr_μ of Gr_G lie in the identity or the non-identity component depending on the parity of $\mu \in \mathbf{Z}_{\geq 0}$. In this way we say that the IC-sheaves of Gr_{2n} agree with the simple objects of $\mathrm{Gr}_{\mathrm{SL}_2}$, corresponding to V_{2n} under the geometric Satake, and the IC-sheaves of Gr_{2n+1} are simple objects in the other component, corresponding to V_{2n+1} that are SL_2 -representations that don't descend to a PGL_2 -representation. Note that by Gr_n (instead of $\mathrm{Gr}_{\leq n}$, for simplicity) we will mean the closure of Gr_μ where μ is the cocharacter indexed by n . Anyway, apart from recalling some basics, this paragraph is meant to convince the audience that working with PGL_2 is not that different from SL_2 and usually contains slightly more information.

4.1. Equivariant cohomology. Let's first try to understand the equivariant cohomology functor

$$\mathrm{Rep}(\check{G}) \rightarrow \mathrm{Mod}_{H_{L+G \times \mathbf{G}_m}^*(\mathrm{Gr}_G)} \simeq \mathrm{QCoh}(N_{(\mathfrak{t}/W)^2} \Delta \sqcup N_{(\mathfrak{t}/W)^2} \Delta), \quad V \mapsto H_{L+G \times \mathbf{G}_m}(\mathrm{Gr}_G, \mathbf{S}(V)).$$

In our special case, it is known that Gr_n 's are rationally smooth and $H_{L+G \times \mathbf{G}_m}^*(\mathbf{S}(V_n)) \simeq H_{L+G \times \mathbf{G}_m}^*(\mathrm{Gr}_n)$ as a $H_{L+G \times \mathbf{G}_m}^*(\mathrm{Gr})$ -module. Ginzburg [19] computed the nonequivariant cohomology $H^*(\mathrm{Gr}_n)$ and showed that it's a cyclic $H^*(\mathrm{Gr})$ -module (i.e., generated by a single element), then by graded Nakayama lemma, $H_{L+G \times \mathbf{G}_m}^*(\mathrm{Gr}_n)$ is cyclic as a $H_{L+G \times \mathbf{G}_m}^*(\mathrm{Gr})$ -module. Since

$$H_{L+G \times \mathbf{G}_m}^*(\mathrm{Gr}) \simeq \mathbf{C}[N_{(\mathfrak{t}/W)^2} \Delta \sqcup N_{(\mathfrak{t}/W)^2} \Delta],$$

$H_{L+G \times \mathbf{G}_m}^*(\mathrm{Gr}_n)$ must be the structure sheaf of a closed subscheme

$$A_n \subset N_{(\mathfrak{t}/W)^2} \Delta$$

sitting in one of the two copies of $N_{(\mathfrak{t}/W)^2} \Delta$ depending on the parity of n . Now the task is to pin down this subscheme A_n , which we now do explicitly.

Construction 4.1 (Subschemes Γ_n and A_n explicitly). Use $\omega : \check{T} \rightarrow \mathbf{G}_m$ to denote a generator of $\mathbf{X}^*(\check{T})$ as a \mathbf{Z} -module, which also gives an element $\omega \in \mathfrak{t}$. Then $P_n = \{n\omega, (n-2)\omega, \dots, (2-n)\omega, -n\omega\}$ is the set of weights of V_n . Recall

$$\Gamma_i := \{(x_1, x_2, a) \in \mathfrak{t} \times \mathfrak{t} \times \mathbf{A}^1 : x_2 = x_1 + i a \omega\} \subset \mathfrak{t} \times \mathfrak{t} \times \mathbf{A}^1, \quad i = -n, -n+2, \dots, n-2, n.$$

Let $\Gamma(n) \subset \mathfrak{t} \times \mathfrak{t} \times \mathbf{A}^1$ denote the closed subscheme cut out by the product of all above equations. Consider the subscheme $(\pi, \pi, \mathrm{id})\Gamma(n) \subset \mathfrak{t}/W \times \mathfrak{t}/W \times \mathbf{A}^1$. Define A_n to be the scheme-theoretic preimage of $(\pi, \pi, \mathrm{id})\Gamma(n)$ in $N_{(\mathfrak{t}/W)^2} \Delta$ along the natural map

$$N_{(\mathfrak{t}/W)^2} \Delta \rightarrow (\mathfrak{t}/W)^2 \times \mathbf{A}^1.$$

To see why this gives the correct construction, note that $\mathcal{H} := H_{L+G \times \mathbf{G}_m}^*(\mathbf{S}(V))$ is a flat $\mathbf{C}[\mathfrak{t}/W \times \mathbf{A}^1]$ -module, and there is a Cartesian square

$$\begin{array}{ccc} \eta \times \mathfrak{t}/W & \xlongequal{\quad} & \eta \times \mathfrak{t}/W \\ \downarrow & & \downarrow \\ N_{(\mathfrak{t}/W)^2} \Delta & \longrightarrow & (\mathfrak{t}/W)^2 \times \mathbf{A}^1. \end{array}$$

where η denotes the generic fiber of $\mathfrak{t}/W \times \mathbf{A}^1$. One may show that to check whether two cyclic modules over $N_{(\mathfrak{t}/W)^2} \Delta$ that are both flat over $\mathfrak{t}/W \times \mathbf{A}^1$ are isomorphic, it suffices to check over the generic fiber of $\mathfrak{t}/W \times \mathbf{A}^1$. On the generic fiber, the result follows from the generic splitting of the canonical filtration.

To be even more explicit, let $v_i \in H^{i+n}(\mathrm{Gr}_n)$ denote the fundamental class of $\mathrm{Gr}_n \cap T_i$, which is a closed irreducible subvariety of Gr_n of dimension $(n-i)/2$, for $i = -n, -n+2, \dots, n-2, n$. Then these v_i 's generate

²It is a general fact [check] that if A is a B -algebra where both A and B are integral domains, then if M and N are two cyclic A -modules that are flat over B and if $M_K \simeq N_K$ where K is the fraction field of B , then M and N are isomorphic.

$H^*(\text{Gr}_n)$ as a \mathbf{C} -vector space. The G -action on Gr_n induces an action of \mathfrak{sl}_2 on $H^*(\text{Gr}_n)$. The actions of $e, h, f \in \mathfrak{sl}_2$ can be explicitly written in terms of the basis

$$hv_i = iv_i, \quad ev_{i-2} = \frac{n+i}{2}v_i, \quad fv_{i+2} = \frac{n-i}{2}v_i.$$

We have shown that as a $H_{T \times \mathbf{G}_m}^*(\text{Gr})$ -module, $H_{T \times \mathbf{G}_m}^*(\text{Gr}_n)$ is cyclic and hence generated by \tilde{v}_{-n} the fundamental class of $\text{Gr}_n = \text{Gr}_n \cap \overline{T_{-n}}$. Consider the inclusion

$$\iota_i : \text{Gr}_n \cap T_i \hookrightarrow \text{Gr}_n.$$

Using the fact that $\text{Gr}_n \cap S_i$ is a transversal slice to $\text{Gr}_n \cap \overline{T_i}$ in Gr_n and that the action of $T \times \mathbf{G}_m$ acts on $\text{Gr}_n \cap S_i = \mathbf{A}^{(n+i)/2}$ linearly with weights $x + (i-1)\hbar, x + (i-2)\hbar, \dots, x + (i-n)/2 \cdot \hbar$, we may deduce that

$$\iota_i^! \tilde{v}_{-n} = (x + (i-1)\hbar)(x + (i-2)\hbar) \cdots (x + \frac{i-n}{2}\hbar) \cdot \iota_i^* \tilde{v}_{-n} \in H_{T \times \mathbf{G}_m}^*(\text{Gr}_n \cap T_i).$$

Using this we may show that

$$\tilde{v}_i := (x + (i-1)\hbar)(x + (i-2)\hbar) \cdots (x + \frac{i-n}{2}\hbar) \cdot \tilde{v}_{-n}$$

has image v_i in $H^*(\text{Gr}_n)$ under the natural map

$$H_{T \times \mathbf{G}_m}^*(\text{Gr}_n) \rightarrow H^*(\text{Gr}_n)$$

and \tilde{v}_i moreover is a generator of the i -th filtered piece of $H_{T \times \mathbf{G}_m}^*(\text{Gr}_n)$.

In this way, we understood how to compare the canonical filtration with the cyclic $\mathbf{C}[N_{(\mathfrak{t}/W)^2}\Delta]$ -structure.

4.2. Kostant functor. On the dual side, again by passing to the generic fiber and flatness, we may prove that

$$\phi(V_n) \simeq \mathcal{O}_{A_n}$$

as a $\mathbf{C}[(\mathfrak{t}/W)^2 \times \mathbf{A}^1]$ -module. Here \mathcal{O}_{A_n} is naturally a sheaf on the deformation to the normal cone and then we may push forward to $(\mathfrak{t}/W)^2 \times \mathbf{A}^1$.

As a $\mathbf{C}[\hbar]$ -module, V_n is free with a basis $\{v_n, v_{n-2}, \dots, v_{-n}\}$ such that e, h, f acts by

$$hv_i = i\hbar v_i, \quad ev_{i-2} = \frac{n+i}{2}\hbar v_i, \quad fv_{i+2} = \frac{n-i}{2}\hbar v_i.$$

The universal Verma module $\mathcal{M}_{\hbar}(-1)$ is a free $\mathbf{C}[\hbar, x]$ -module with a basis $\{m_{-1}, m_{-3}, \dots\}$ where the actions of e, h, f are given by

$$hm_i = (x + i\hbar)m_i \quad em_i = \frac{-i-1}{2}\hbar(x - \frac{i+1}{2}\hbar)m_{i+2}, \quad fm_i = m_{i-2}.$$

We constructed a canonical filtration

$$0 = F^{n+2} \subset F^n \subset \dots \subset F^{2-n} \subset F^{-n} = \mathcal{M}_{\hbar}(-1) \otimes_{\mathbf{C}[\hbar]} V_n$$

by U_{\hbar} -submodules with desired graded pieces

$$F^i/F^{i+2} \simeq \mathcal{M}_{\hbar}(i-1) \otimes_{\mathbf{C}} V_{n,i}$$

where $V_{n,i}$ is the one-dimensional \mathbf{C} -(vector space) generated by v_i . There is a unique vector

$$s_i \in (\mathcal{M}_{\hbar}(-1) \otimes_{\mathbf{C}} V_n) \otimes_{\mathbf{C}[\hbar, x]} \mathbf{C}(\hbar, x)$$

such that $es_i = 0$ and that $s_i \equiv m_{-1} \otimes v_i$ modulo $U_{\hbar}(s_{i+2}, s_{i+4}, \dots, s_n)$ by which we denote the U_{\hbar} -submodule of $(\mathcal{M}_{\hbar}(-1) \otimes_{\mathbf{C}} V_n) \otimes_{\mathbf{C}[\hbar, x]} \mathbf{C}(\hbar, x)$ generated by s_{i+2}, \dots, s_n . Let \bar{s}_i denote the image of s_i in the ψ -covariant

$$((\mathcal{M}_{\hbar}(-1) \otimes_{\mathbf{C}} V_n) \otimes_{\mathbf{C}[\hbar, x]} \mathbf{C}(\hbar, x)) / (f-1) \simeq \varphi(V_n) \otimes_{\mathbf{C}[\hbar, x]} \mathbf{C}(\hbar, x).$$

In this way, we produce canonical generators of the graded pieces F^i/F^{i+2} .

The cyclic module structure demands that $\varphi(V_n)$ is generated by $\overline{m_1 \otimes v_{-n}}$ as a $\mathbf{C}[N_{(\mathfrak{t}/W)^2}\Delta]$ -module. We may express $\overline{m_1 \otimes v_{-n}}$ in terms of the basis we produced above

$$\overline{m_1 \otimes v_{-n}} = \sum_{i=-n}^n (x + (i-1)\hbar)^{-1} (x + (i-2)\hbar)^{-1} \cdots (x + \frac{i-n}{2}\hbar)^{-1} \bar{s}_i.$$

5. PROOF SKETCH OF THE MAIN THEOREMS

In this section, we sketch (very roughly!) the proof of the main theorems.

5.1. Step 1: identify the topological and the algebraic. We first want to show that the following diagram commutes.

$$\begin{array}{ccc}
 & \text{Rep}(\check{G}) & \\
 \text{Fr} \swarrow & & \searrow \mathbf{S} \\
 \mathcal{H}\mathcal{C}_{\hbar}^{\text{fr}} & \xrightarrow{\phi} & \text{Perv}_{L+G \times \mathbf{G}_m}(\text{Gr}_G) \\
 \kappa_{\hbar} \downarrow & & \downarrow \mathcal{H}_{L+G \times \mathbf{G}_m}^* \\
 \text{Coh}(N_{(\mathfrak{t}/W)^2} \Delta) & \xrightarrow{\simeq} & \text{Mod}_{H_{L+G \times \mathbf{G}_m}^*}(\text{Gr}_G)
 \end{array}$$

But as we know, saying a diagram of categories commute is ambiguous, as here we are essentially saying two functors are isomorphic, and we would like to specify this isomorphism and say that it has good compatibility properties. Now let's make this precise.

Theorem 5.1. *Let $V \in \text{Rep}(\check{G})$. There is a unique isomorphism of $\mathbf{C}[\mathfrak{t}/W \times \mathfrak{t}/W \times \mathbf{A}^1]$ -modules*

$$\eta_V : H_{L+G \times \mathbf{G}_m}^*(\text{Gr}_G, \mathbf{S}(V)) \simeq \phi(V)$$

such that $(\pi, \text{id}, \text{id})^* \eta_V$

- (1) *preserves the canonical filtrations on two sides, and*
- (2) *induces the identity isomorphism of the associated graded of two sides when base changed along $(\pi, \text{id}, \text{id})$ that are both isomorphic to*

$$(\text{id}, \pi, \text{id})_* \bigoplus_{\lambda} \mathcal{O}_{\Gamma_{\lambda}} \otimes_{\mathbf{C}} V_{\lambda}.$$

Remark 5.2. This isomorphism is moreover compatible with the monoidal structures.

Proof sketch. Since the canonical filtrations on both sides split after passing to the generic fiber of $\mathfrak{t} \times \mathbf{A}^1$, we are able to identify $(\pi, \text{id}, \text{id})^* H_{L+G \times \mathbf{G}_m}^*(\text{Gr}_G, \mathbf{S}(V))$ and $(\pi, \text{id}, \text{id})^* \phi(V)$ both as $\mathbf{C}[\mathfrak{t} \times \mathfrak{t}/W \times \mathbf{A}^1]$ -submodules of

$$\text{gr}_{\eta} := \left((\text{id}, \pi, \text{id})_* \bigoplus_{\lambda} \mathcal{O}_{\Gamma_{\lambda}} \otimes_{\mathbf{C}} V_{\lambda} \right) \otimes_{\mathbf{C}[\mathfrak{t} \times \mathbf{A}^1]} \mathbf{C}(\mathfrak{t} \times \mathbf{A}^1)$$

both equipped with W -actions.

We may first show that the induced W -actions on gr_{η} are the same. The fact can be checked explicitly for the rank 1 case following our rank 1 computations, and for the general case, we may pass to subminimal Levi subgroups L that have only one nontrivial root and use the transitivity and compatibility of various constructions with passing to Levi subgroups. The identification of $(\pi, \text{id}, \text{id})^* H_{L+G \times \mathbf{G}_m}^*(\text{Gr}_G, \mathbf{S}(V))$ and $(\pi, \text{id}, \text{id})^* \phi(V)$ as $\mathbf{C}[\mathfrak{t} \times \mathfrak{t}/W \times \mathbf{A}^1]$ -submodules then also follows from Levi subgroup considerations. \square

5.2. Step 2: proofs of Theorem 1.10 and Theorem 1.16 by Ginzburg's computations. Let $\mathcal{J}\mathcal{C}$ and $\widetilde{\mathcal{J}\mathcal{C}}$ denote the full subcategories of semisimple complexes in $\mathcal{D}_{L+G \times \mathbf{G}_m}(\text{Gr}_G)$ and $\mathcal{D}_{L+G}(\text{Gr})$. Theorem 1.10 and Theorem 1.16 follow from the following result, due to Ginzburg.

Lemma 5.3. *The functors*

$$\mathcal{H}_{L+G \times \mathbf{G}_m} : \mathcal{J}\mathcal{C} \rightarrow \text{Coh}_{\mathbf{G}_m}(N_{(\mathfrak{t}/W)^2} \Delta), \quad \mathcal{H}_{L+G}^* : \widetilde{\mathcal{J}\mathcal{C}} \rightarrow \text{Coh}_{\mathbf{G}_m}(\mathbf{T}(\mathfrak{t}/W))$$

are fully faithful.

5.3. **Step 3: conclude by purity.** We shall construct a monoidal functor

$$\Psi : \mathcal{D}_{\text{perf}, \check{G}}(U_{\hbar}^{\square}) \rightarrow \mathcal{D}_{L+G \times \mathbf{G}_m}(\text{Gr}_G)$$

extending \mathcal{S} . This suffices because then Ψ sends a set of generators to a set of generators and induces an isomorphism of dg categories between generators, and hence is an equivalence.

It suffices to construct a compatible collection

$$\Psi_{\lambda} : \mathcal{D}_{\text{perf}, \check{G}}(U_{\hbar}^{\square})_{\leq \lambda} \rightarrow \mathcal{D}_{L+G \times \mathbf{G}_m}(\text{Gr}_{\leq \lambda})$$

where by $\mathcal{D}_{\text{perf}, \check{G}}(U_{\hbar}^{\square})_{\leq \lambda}$ we mean the full subcategory in $\mathcal{D}_{\text{perf}, \check{G}}(U_{\hbar}^{\square})$ generated by the objects $V \otimes U_{\hbar}^{\square}$ where V is an irreducible representation of \check{G} with highest weight $\mu \leq \lambda$.

To conclude, we need some formal inputs from the theory of purity of perverse sheaves. We won't explain the standard procedure of passing to the characteristic p . Let X be an algebraic variety over \mathbf{F}_q and let $\mathcal{D}(X_{\overline{\mathbf{F}}_q})$ be the bounded constructible ℓ -adic derived category of $X_{\overline{\mathbf{F}}_q}$ where $\ell \neq p$. Let \mathcal{F} be a pure of weight 0 object of $\mathcal{D}(X)$ of geometric origin. The space

$$\text{Ext}^i(\mathcal{F}_{\overline{\mathbf{F}}_q}, \mathcal{F}_{\overline{\mathbf{F}}_q}) = \bigoplus_j \text{Ext}_j^i(\mathcal{F}_{\overline{\mathbf{F}}_q}, \mathcal{F}_{\overline{\mathbf{F}}_q})$$

carries a canonical grading by Frobenius weights. Let \mathcal{E} be a graded algebra and $\phi : \mathcal{E} \rightarrow \text{Ext}^{\bullet}(\mathcal{F}_{\overline{\mathbf{F}}_q}, \mathcal{F}_{\overline{\mathbf{F}}_q})^{\text{op}}$ be a morphism sending \mathcal{E}^i to $\text{Ext}_i^i(\mathcal{F}_{\overline{\mathbf{F}}_q}, \mathcal{F}_{\overline{\mathbf{F}}_q})$. View \mathcal{E} as a dg algebra with zero differentials.

Lemma 5.4. *There exists a canonical functor*

$$\Phi_X : \mathcal{D}_{\text{perf}}(\mathcal{E}) \rightarrow \mathcal{D}(X_{\overline{\mathbf{F}}_q})$$

sending the free module to $\mathcal{F}_{\overline{\mathbf{F}}_q}$ and inducing the map ϕ on Ext groups.

We have an isomorphism

$$\text{End}^{\bullet} \left(\bigoplus_{\mu \leq \lambda} V_{\mu} \otimes_{\mathbf{C}} U_{\hbar}^{\square} \right) \simeq \text{End}^{\bullet} \left(\bigoplus_{\mu \leq \lambda} \text{IC}_{\mu} \right).$$

By ‘‘purity of equivariant Ext’s between IC sheaves on the affine Grassmannians,’’ the above lemma applies to provide us with the functor Ψ_{λ} .

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