Notation F = local field, $\mathcal{O}$ = ring of integers, k = finite residue field
$\pi \in \mathcal{O}$ unif., $\mathfrak{p} = (\pi)$

$G/\mathcal{O}$ split red gp, $B = NA \subset G$ Borel, $\bar{B} = \bar{N}A$ opp. Borel,
assume $[\mathfrak{g}, \mathfrak{g}]$ simple

W = Weyl gp of G

Def The Iwahori subgp of G is

$$I = \{ g \in G(\mathcal{O}) : g \bmod \pi \in \bar{B}(k) \subset G(k) \}$$

Note I used opposite convention in my talk but $\bar{B}$ is more consistent w/ the literature

e.g. $G = GL_2$, $I = \begin{pmatrix} \mathcal{O}^\times & \pi\mathcal{O} \\ \mathcal{O} & \mathcal{O}^\times \end{pmatrix} \subset G(F)$

Today study $\mathcal{H}_I := \mathbb{C}[I \backslash G(F) / I]$
which is a non-comm. alg under convolution
$\mathrm{vol}(I) = 1$

Sources Iwahori-Matsumoto, "On some Bruhat decomposition..."
Chan-Savin, "Iwahori component of the Gelfand-Graev repn"

Lusztig, "Affine Hecke algebras and their graded versions"
Achar & Riche, "Central sheaves on affine flag varieties" §5.1
Chriss & Ginzburg "Repn theory & cx geometry" §7.1

__Recall__ (extended) affine Weyl gp

$\widetilde{W} := X_*(A) \rtimes W$

$X_*(A) \subset G(F)$ by $\lambda \mapsto \pi^\lambda = \lambda(\pi)$

__Def__ For $\tilde{\omega} = \lambda w \in \widetilde{W}$

$$\ell(\tilde{\omega}) := \sum_{\substack{\alpha \in \Delta^+ \\ w(\alpha) \in \Delta^+}} |\langle \lambda, w\alpha \rangle| + \sum_{\substack{\alpha \in \Delta^+ \\ w(\alpha) \in \Delta^-}} |\langle \lambda, w\alpha \rangle + 1|$$

__Obs__ (1) ℓ restricts to usual length fn on W

(2) if $\tilde{\omega} = \lambda$ dominant, $\ell(\tilde{\omega}) = \langle \lambda, 2\rho \rangle$

(3) if $R \subseteq X_*(A)$ is lattice spanned by coroots

then $\ell(\tilde{\omega}) = 0 \Rightarrow \tilde{\omega} = 1$ for $\tilde{\omega} \in R \rtimes W$

$X_*(A)/R = \pi_1(G) \rightsquigarrow$ let's assume $\pi_1(G) = 1$ for simplicity

in general, carry around $\Omega = X_*(A)/R \longleftrightarrow \{$elts $\tilde{\omega} \in \widetilde{W}$ s.t. $\ell(\tilde{\omega}) = 0\}$

__e.g.__ $G = PGL_2$

$\cancel{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}}$ ~~has~~ $\cancel{\ell(\tilde{\omega}) = 0}$

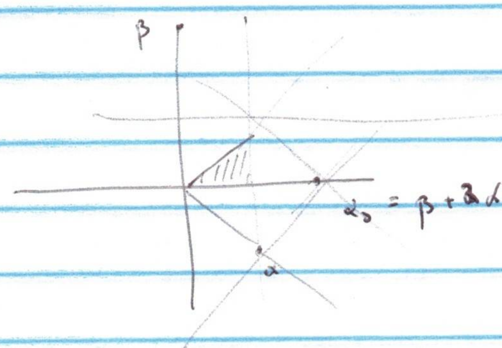
$\tilde{\omega} = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ has $\ell(\tilde{\omega}) = 0$

Geometric picture (from Iwahori-Matsumoto)

For $\alpha \in \Delta$, $k \in \mathbb{Z}$ $\quad P_{\alpha,k} = \{ \langle \alpha, x \rangle = k \} \subseteq \mathfrak{a}_{\mathbb{R}}^*$

$\tilde{W}$ = gp gen by reflections across $P_{\alpha,k}$
(uses semisimplicity!)

e.g. $G = Sp_4$



fundamental domain for $\tilde{W}$ = $D_0 = \{ x \in \mathfrak{a}_{\mathbb{R}}^* : (\alpha, x) > 0 \ \forall \alpha \in \Delta^+$
$(\alpha_0, x) < 1 \}$

(a chamber in $\mathfrak{a}_{\mathbb{R}}^* - \cup P_{\alpha,k}$)

α_0 := highest root

Prop ① $\ell(\tilde{w}) = \#$ of $P_{\alpha,k}$ s.t. $\tilde{w} D_0$ & D_0 are on opposite sides of $P_{\alpha,k}$

② $\ell(\tilde{w}) = 1 \iff \tilde{w} \in S := \{\underbrace{s_{\alpha_1}, \dots, s_{\alpha_\ell}}_{\text{reflections across simple roots}}, \underset{\text{reflection along } P_{\alpha_0,1}}{(t\alpha_0^\vee) s_{\alpha_0}}\}$

③ $\ell(\tilde{w}) =$ length of shortest expression
$\tilde{w} = s_1 \dots s_r$, $s_i \in S$

I won't prove this but you can get it from contemplating the geometric picture.

Prop (Bruhat Lemma) Let $w \in \tilde{W}$, $s \in S$, then either:

① $\ell(sw) = \ell(w) + 1$, $IsIwI = IswI$ or

② $\ell(sw) = \ell(w) - 1$, $IsIwI = IswI \cup IwI$

dichotomy follows from prop

Pf $s = s_\alpha \in W$ (other case similar but analyzed separately) Pf from Iwahori-Matsumoto.

~~Since $I = N(\mathcal{O}) A(\mathcal{O}) \bar{N}(\mathfrak{p})$ $I =$~~

$Is_\alpha I = Is_\alpha \bar{N}(\mathfrak{p}) A(\mathcal{O})$ ~~$N(\mathcal{O})$~~

~~$= Is_\alpha N(\mathcal{O})$~~

~~$= Is_\alpha X_\alpha(\mathcal{O})$~~

~~$X_\alpha : G_a \to G$ root gp hom.~~

~~$\therefore IsIwI = Is_\alpha X_\alpha(\mathcal{O}) wI = I s_\alpha w\, w^{-1} X_\alpha(\mathcal{O}) w I$~~

since $\quad I = \bar{N}(\mathcal{O})\, A(\mathcal{O})\, N(\mathfrak{p})$

$$I s_\alpha I = I s_\alpha N(\mathfrak{p}) A(\mathcal{O}) \bar{N}(\mathcal{O})$$
$$= I s_\alpha \bar{N}(\mathcal{O})$$
$$= I s_\alpha X_{-\alpha}(\mathcal{O})$$

$\therefore I s_\alpha I w I =$
$$I s_\alpha X_{-\alpha}(\mathcal{O}) w I = I s_\alpha w\, w^{-1} X_{-\alpha}(\mathcal{O}) w I$$

For ① $\quad$ STS if $\ell(s_\alpha w) = \ell(w)+1$ $\quad$ that $w^{-1} X_{-\alpha}(\mathcal{O}) w \subseteq I$

prove by contemplating length formula

For ② $\quad$ put $\quad w = s_\alpha \tau \quad \ell(\tau) = \ell(w)-1$

then $\quad I s_\alpha I s_\alpha \tau I = I s_\alpha I s_\alpha I \tau I$ by ①

$\therefore$ STS $\quad I s_\alpha I s_\alpha I = I s_\alpha I \cup I$

as above $\quad I s_\alpha I s_\alpha I = I s_\alpha X_{-\alpha}(\mathcal{O}) s_\alpha I$
$$= I X_{+\alpha}(\mathcal{O}) I$$
$$= I X_{+\alpha}(\mathfrak{p}) I \cup I X_\alpha(\mathcal{O}^\times) I$$

clearly $I X_{+\alpha}(\mathfrak{p}) I = I$

For $t \in \mathcal{O}^\times$ $\quad$ claim $\quad I X_\alpha(t) I = I s_\alpha I$

reduces to SL_2 computation:

$$\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -t & 0 \\ 1 & -t^{-1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

§ Structure of $\mathcal{H}_I$

Cor (of Bruhat Lemma) $I \backslash G / I = \tilde{W}$

pf sketch B. L. $\Rightarrow$ $\bigcup_{w \in \tilde{W}} I w I$ is a subgp

of $G(F)$, contains, I, $A(F)$, W, hence all of $G(F)$

Need $I \sigma I = I \tau I \Rightarrow \sigma = \tau$ for $\sigma, \tau \in \tilde{W}$;

induct on $\min\{\ell(\sigma), \ell(\tau)\}$

Hence $\mathcal{H}_I$ has a natural basis $T_x := \mathbb{1}_{IxI}$

for $x \in \tilde{W}$

Prop We have the following relations in $\mathcal{H}_I$:

① If $x, y \in \tilde{W}$ w/ $\ell(xy) = \ell(x) + \ell(y)$ then $T_x T_y = T_{xy}$

② If $s \in S$ then $(T_s + 1)(T_s - q) = 0$

Note This determines alg. structure, & implies all T_x invertibly

Pf ① wlog $x = s \in S$

$$T_s T_y(g) = \mathrm{vol}\{h : gh^{-1} \in IsI,\ h \in IyI\}$$

$\therefore$ $T_s T_y$ supp on $g \in IsI \cdot IyI = IsyI$

$$T_s T_y(sy) = \mathrm{vol}\{IsI\,sy \cap IyI\} =: H$$

$H \supseteq Iy$, WTS $H = Iy$

indeed $IsI\,sy \subseteq IsIsIy = IsIy \cup Iy$

so STS $IsIy \cap IyI = \emptyset$, clear b/c $IsIyI = IsyI$

② $T_s T_s(g) = \mathrm{vol}\{h : gh^{-1} \in IsI,\ h \in IsI\}$

$\therefore$ $T_s T_s$ supp on $g \in IsI \cdot IsI = IsI \cup I$

for $g = 1$: $T_s T_s(1) = \mathrm{vol}(IsI) = q$

for $g = s$: $T_s T_s(s) = \mathrm{vol}\{h : h^{-1} \in IsIs,\ h \in IsI\}$

$= \mathrm{vol}(IsI \cap sIsI)$

Note $sIsI \subseteq sIsIsI = I \cup IsI$

so $\mathrm{vol}(IsI \cap sIsI) = \mathrm{vol}(sIsI \setminus I)$

$= \mathrm{vol}(sIsI) - \mathrm{vol}(I) = q - 1$

$\therefore$ $T_s T_s = (q-1)T_s + q$ ✓

The affine Hecke alg.

Prop $\exists!$ $\mathbb{Z}[q, q^{-1}]$ - alg $\tilde{\mathcal{H}}$, free over $\mathbb{Z}[q, q^{-1}]$ on the generators T_w for $w \in \tilde{W}$, & satisfying

① $T_x T_y = T_{xy}$ when $\ell(xy) = \ell(x) + \ell(y)$

② $(T_s + 1)(T_s - q) = 0$

Called the affine Hecke alg

Prop may be true over $\mathbb{Z}[q]$ but specializing @ $q=0$ does not seem to be useful?

Note that $\mathcal{H}_I = \tilde{\mathcal{H}} \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{C}$

So $\tilde{\mathcal{H}}$ lets you treat q as a variable - ~~[illegible]~~ which is ~~[illegible]~~ useful in practice, as we will see.

Def (Bernstein) For any $\lambda \in X_*(A)$,

$$\Theta_\lambda := q^{-\langle \lambda, \rho \rangle} T_{\lambda_1} T_{\lambda_2}^{-1} \in \tilde{\mathcal{H}} \text{ or } \mathcal{H}_I$$

where λ_1, λ_2 are dominant s.t. $\lambda = \lambda_1 - \lambda_2$

Well-def'd since $T_{\lambda_1} T_{\lambda_2} = T_{\lambda_1 + \lambda_2}$ for λ_1, λ_2 dominant,

which also shows Θ_λ all commute

Note: need $q^{1/2}$ coeffs in non-ss case

Prop ① For $s = s_\alpha$ a simple reflection,

$$T_s \Theta_\lambda - \Theta_{s(\lambda)} T_s = (q-1) \frac{\Theta_\lambda - \Theta_{s(\lambda)}}{1 - \Theta_{-\alpha^*}}$$

② The elts $T_w \Theta_\lambda$, for $\lambda \in X_*(A)$ & $w \in W$, form a basis of $\mathcal{H}_I$ (or $\tilde{\mathcal{H}}$)

Pf sketch (From Lusztig) (1) use linearity to reduce to $\langle \lambda, \alpha \rangle = 1$ or 0 & λ dominant

if $\langle \lambda, \alpha \rangle = 0$ need $T_s T_\lambda = T_\lambda T_s$ which is true b/c

$\lambda s = s\lambda$ & $\ell(\lambda s) = \ell(\lambda)+1$

if $\langle \lambda, \alpha \rangle = 1$ need $q \Theta_\lambda = T_s \Theta_{s(\lambda)} T_s$

which reduces to $T_s^{-1} T_\lambda T_s^{-1} T_\lambda \overset{?}{=} T_{s(\lambda)+\lambda}$

b/c $\lambda + s(\lambda)$ is dominant

note $\ell(\lambda + s(\lambda)) = 2\ell(\lambda) - 2$

$\ell(\lambda s) = \ell(\lambda) - 1$

$\Rightarrow T_s T_{\lambda s} = T_\lambda$ ie $T_s^{-1} T_\lambda = T_{\lambda s}$

and $T_{\lambda s} T_{\lambda s} = T_{\lambda s \lambda s} = T_{\lambda + s(\lambda)}$ ✓

(2) To show they span: let $\mathcal{H}_\Theta \subseteq \mathcal{H}_I$ (or $\tilde{\mathcal{H}}$) be linear span of $T_w \Theta_\lambda$'s

playing w/ relations shows $\mathcal{H}_\Theta$ is a subalgebra

$\therefore$ STS it contains T_s $\forall s \in S$

ie STS it contains $T_{\alpha_0^* s_{\alpha_0}}$

True b/c $\ell(s_{\alpha_0}) = \ell(\alpha_0^*) - 1$

$\Rightarrow T_{\alpha_0^* s_{\alpha_0}} T_{s_{\alpha_0}} = T_{\alpha_0^*}$

$\Rightarrow T_{\alpha_0^* s_{\alpha_0}} = T_{s_{\alpha_0}}^{-1} T_{\alpha_0^*}$

& $T_{s_{\alpha_0}}^{-1}, T_{\alpha_0^*} \in \mathcal{H}_\Theta$

To show independence: if

$$\sum c_i T_{w_i} \Theta_{\lambda_i} = 0$$ then for $x >> 0$ $\lambda_i + x$ all dominant

& $$\sum c_i T_{w_i} \Theta_{\lambda_i + x} = 0$$

so STS $T_w T_\lambda$ are lin ind. for $w \in W$, λ dominant

but then $\ell(w\lambda) = \ell(w) + \ell(\lambda)$

so $T_w T_\lambda = T_{w\lambda}$ are ind.

Thm (Bernstein) the center of $\mathcal{H}_I$ or $(\tilde{\mathcal{H}})$ is spanned by

$$\sum_{\substack{\lambda' = w \cdot \lambda \\ w \in W}} \Theta_{\lambda'} \quad \text{for } \lambda \text{ dominant}$$

Pf Clear these lie in the center from Bernstein's relation.

Let $R \subseteq \tilde{\mathcal{H}}$ be the span of these elts, $Z \subseteq \tilde{\mathcal{H}}$ the true center.

Note $\tilde{\mathcal{H}}/(q-1) = \mathbb{Z}[\tilde{W}]$ whose center is $\mathbb{Z}[X_*(A)]^W$
$= R/(q-1)$

since $Z \cap (q-1)\tilde{\mathcal{H}} = (q-1)Z$,
$Z/_{(q-1)Z} = R/(q-1)R$

Nakayama $\Rightarrow$ $Z_m = R_m$ where $m = (q-1)$
Since $\tilde{\mathcal{H}}$ is free over $\mathbb{Z}[q, q^{-1}]$, this implies $Z = R$.

§ The aspherical module

$\mathcal{H} = \tilde{\mathcal{H}}$ or $\mathcal{H}_I$, $\mathcal{H}_W \subset \mathcal{H}$ = subalg gen by T_w for $w \in W$

Note $\mathcal{H} \cong \mathcal{A} \otimes \mathcal{H}_W$
over $\mathbb{C}$ or $\mathbb{Z}[q, q^{-1}]$
where $\mathcal{A}$ = span of Θ_λ's = $\mathbb{C}[X_*(A)]$
or $\mathbb{Z}[q, q^{-1}]\,[X_*(A)]$

Def $M_{asp} := \mathcal{H} \otimes_{\mathcal{H}_W} \mathrm{sgn}$

ie free of rk 1 over $\mathcal{A}$, $\mathcal{H}_W$ acts by
sgn repr $T_s = -1$ $\forall$ s simple reflection in W

Note $\mathrm{Hom}_{\mathcal{H}}(\mathcal{H} \otimes_{\mathcal{H}_W} \mathrm{sgn}, M) = \mathrm{Hom}_{\mathcal{H}_W}(\mathrm{sgn}, M)$

Two other characterizations given in Arkhipov-Bezrukavnikov

(1) $M_{asp} = \mathcal{H} / \langle \underline{H}_w, w \notin X \rangle$
minimal length reps
of left cosets of $W \subseteq \tilde{W}$

(2) $M_{asp} = (\text{c-Ind}_{N(F)}^{G(F)} \Psi)^I$ where $\Psi: N(F) \to \mathbb{C}$
is trivial on $N(\mathfrak{p})$ but ~~gen~~ induced
char on $N(k) = N(\mathcal{O})/N(\mathfrak{p})$ is generic

following Chan & Savin

We will prove ②. First step: check V^I free of rk one / $\mathcal{A}$ where $V := \text{c-Ind}_{N(F)}^{G(F)} \Psi$; then compute T_w action.

By a general thm on I-fixed vectors,

$$V^I \cong V_{\bar{N}}^{A(\mathcal{O})} \quad \text{as } \mathcal{A} = \mathbb{C}[X_*(A)] \text{-module}$$

induced by $V \to V_{\bar{N}}$

Prop $V_{\bar{N}} = (V_0)_{\bar{N}}$ where $V_0 \subseteq V$ is subspace of fns supp on $NA\bar{N}$

Pf For $w \in W$ consider $X_w = NwA\bar{N}$;

$V_r \subseteq V$ = fns supp in $\bigcup_{\ell(w) \leq r} X_w$

V_w = Whittaker fns on $NwA\bar{N}$ cpctly supp mod N (on the left)

So $0 \to V_{r-1} \to V_r \to \bigoplus_{\ell(w)=r} V_w \quad \forall r \geq 1$

& STS $(V_w)_{\bar{N}} = 0 \quad \forall w \neq 1$

Given $f \in V_w$ supp on $NwA_f\bar{N}_f$, for $A_f \subseteq A(F)$ $N_f \subseteq \bar{N}(F)$ cpct, STS $\exists N_0 \subseteq \bar{N}(F)$ cpct s.t.

$$\int_{N_0} f(w\, a_f n_f n_0)\, dn_0 = 0 \quad \forall \underset{A_f}{a_f} \in,\ \underset{\bar{N}_f}{n_f} \in$$

wlog $a_f, n_f = 1$

since $w \neq 1$ can pick $N_0 \subseteq \bar{N}(F)$ cpct s.t.

$$\int_{wN_0w^{-1} \cap N(F)} \psi(n_0)\, dn_0 = 0$$

$$\therefore \int_{N_0} n_0 f(w)\, dn_0 = \int_{N_0} f(w n_0)\, dn_0 = \int_{N_0} f(wn_0w^{-1} \cdot w)\, dn_0$$

$$= 0$$

$$\text{Prop} \Rightarrow \quad V^I = V_{\bar{N}}^{A(\mathcal{O})} = (V_0)_{\bar{N}}^{A(\mathcal{O})}$$

$$V_0 = \text{fns supp on } NA\bar{N} = C_c^\infty(A\bar{N})$$

$$(V_0)_{\bar{N}} = C_c^\infty(A)$$

$$\text{by} \quad f \longmapsto \left(t \mapsto \int_{\bar{N}(F)} \delta_B(t)^{-1/2} f(tn)\, dn\right)$$

So V^I is free of rk 1 over A, & natural generator is the f supp on $N(F)\, A(\mathcal{O})\, \bar{N}(\mathcal{O})$ satisfying $f(1) = 1$

Step two: Check $T_w f = (-1)^{\ell(w)} f$ for $w \in W$

ie $T_{s_\alpha} f = -f$ for α simple root

Since $G = N\tilde{W}I$ it suffices to show

$T_{s_\alpha} f(w) = -f(w)$ for $w \in \tilde{W}$

Note $T_{s_\alpha} f(w)$ is supp on w s.t. $NwIs_\alpha I \supseteq NI$

which implies $w = 1$ or s_α

(1) $w = s_\alpha$

$$T_{s_\alpha} f(s_\alpha) = \sum_{g \in I s_\alpha I / I} f(s_\alpha g)$$

$$I s_\alpha I = \bigsqcup_{t \in \mathcal{O}/\pi} s_\alpha X_\alpha(t) I$$

so $$T_{s_\alpha} f(s_\alpha) = \sum_{t \in \mathcal{O}/\pi} f(X_\alpha(t) I) = 0 \checkmark$$

(good b/c $f(s_\alpha) = 0$ as well)

(2) $w = 1$

$$T_{s_\alpha} f(1) = \sum_{t \in \mathcal{O}/\pi} f(s_\alpha X_\alpha(t) I)$$

if $t \in (\pi)$ get 0

if $t \notin (\pi)$ we have $s_\alpha X_\alpha(t) \in X_\alpha(-t^{-1}) I$

by the SL_2 identity

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 1 & t \end{pmatrix}$$

$$\therefore T_{s_\alpha} f(1) = \sum_{t \in k^\times} f(X_\alpha(-t^{-1}) I)$$

$$= \sum_{t \in k^\times} \psi(X_\alpha(t))$$

$$= -1 \checkmark \Rightarrow \text{have } \boxed{T_{s_\alpha} f = -f}$$

§ Kazhdan-Lusztig basis

$\exists!$ inv $\iota : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$

sending $q \mapsto q^{-1}$

and $T_w \longmapsto (T_{w^{-1}})^{-1}$ $\forall w \in \tilde{W}$

(Not meaningful for $\mathcal{H}_I$)

<u>Thm</u> (KL) $\forall w \in W$, $\exists!$ $\underline{H}_w \in \tilde{\mathcal{H}}$ s.t.

$\iota(\underline{H}_w) = \underline{H}_w$

and $\underline{H}_w = T_w + \sum_{y<w} q f_y T_y$

w/ $f_y \in \mathbb{Z}[q]$

This gives the KL basis of $\tilde{\mathcal{H}}$ & $\mathcal{H}_I$