SPRINGER AND STEINBERG VARIETIES

MINGJIA ZHANG

We have seen the geometric Satake equivalence, which gives the unramified local Langlands correspondence. Starting this talk, we will move to geometrizations related to the unipotent local Langlands correspondence. Today we will introduce the Springer and Steinberg varieties, which are the main geometric objects on the spectral side. The goal of this talk is to explain the Kazhdan–Lusztig isomorphism, which is the precursor of Arkhipov–Berzrukavnikov, Berzrukavnikov's equivalences. Our main reference is the book [CG97]. Throughout this talk, we will adopt the following notation.

0.1. Notation.

- G is a semisimple, simply connected reductive group over $\mathbb{C}$.
- (B,T) is a Borel pair of G.
- W is the finite Weyl group of G.
- $\mathfrak{g}$ is the Lie algeba of G.
- $\check{G}$, $\check{B}$, $\check{T}$ are the dual groups, considered as algebraic groups over $\mathbb{Z}$.
- $\mathbb{G} = G \times \mathbf{G}_m$, $\mathbb{B} = B \times \mathbf{G}_m$, $\mathbb{T} = T \times \mathbf{G}_m$.
- $R(H)$ for an algebraic group H over $\mathbb{C}$ is the representation ring of H, namely the Grothendieck group of the category of finite dimensional algebraic representations of H.
- $\mathbb{H}_{\check{G}}$ is the extended affine Hecke algebra of $\check{G}$. This is an associative algebra over $R(\mathbf{G}_m) \simeq \mathbb{Z}[q, q^{-1}]$.
- For a scheme X with an action by some algebraic group H, we write X/H for the quotient as a stack and $X /\!\!/ H$ for the coarse moduli space.
- For X being a noetherian algebraic stack, we denote the stable ∞-category of coherent[1] complexes on X by $\mathcal{D}(X)$. All functors on derived categories are derived.

1. Associative algebras from convolution

We would like to study geometrizations of the extended affine Hecke algebra $\mathbb{H}_{\check{G}}$, which is an associative algebra with a maximal commutative subalgebra isomorphic to $R(\mathbb{T})$ and whose center is isomorphic to $R(\mathbb{G})$. We first observe

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[1] We say a quasi-coherent complex $\mathcal{F}$ is coherent if for any smooth morphism $U \to X$ from a locally noetherian affine scheme U, the pullback of $\mathcal{F}$ to U is bounded and has coherent cohomologies.

that one can construct associative algebras using convolution on algebraic varieties, or more generally stacks, in any suitable context. This provides an abundant supply of associative algebras for which one has some control of their commutative subalgebras and centers. Let us review this process.

1.1. **Constructions.** Recall that for a noetherian algebraic stack X, or more generally a noetherian derived algebraic stack, see [Kha22, Section 3], one can define its G-theory spectrum as the algebraic K-theory of the stable ∞-category of coherent complexes on X

$$G(X) := K(\mathcal{D}(X)).$$

To not to use G for both the reductive group we have fixed and the K-theory spectrum, below we will use the notation $K(X)$ instead to denote $K(\mathcal{D}(X))$. It should not be confused with the K-theory spectrum defined using perfect complexes on X, $K(X)$ is more usually the notation for which.

Construction 1.2. Let X and Y be noetherian algebraic stacks over a field k, and $f : Y \to X$ be a proper morphism of finite cohomological dimension. Assume the diagonal map $\Delta_Y : Y \to Y \times_k Y$ has finite tor-amplitude.

Consider the truncated Cech nerve of the map f (where all fiber products are taken in derived algebraic stacks)

$$\begin{tikzcd} Y_3 := Y \times_X Y \times_X Y \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Y_2 := Y \times_X Y \arrow[r, shift left=2] \arrow[r, shift right=2] & Y \arrow[l] \arrow[r, "f"] & X \end{tikzcd}$$

and write p_{12}, p_{23}, p_{13} (respectively p_1, p_2) for the projection maps from the triple self-product of Y (respectively the self-product), and $\Delta_{Y/X}$ for the diagonal embedding $Y \to Y \times_X Y$.

We can define the convolution product $*$ on $\mathcal{D}(Y \times_X Y)$ as

$$\mathcal{F} * \mathcal{G} := p_{13,*}(p_{12}^* \mathcal{F} \otimes p_{13}^* \mathcal{G}).$$

Note that the operations in the formula preserve coherence in our setup. Indeed, we have the following cartesian diagram

$$\begin{tikzcd} Y_3 \arrow[r, "\Delta_{Y_3/Y_2 \times_k Y_2}"] \arrow[d] & Y_2 \times_k Y_2 \arrow[d, "p_2 \times p_1"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_k Y. \end{tikzcd}$$

Using this diagram, we can rewrite the right hand side of the formula[2] for the convolution product as

$$p_{13,*}\Delta^*_{Y_3/Y_2 \times_k Y_2}(\mathcal{F} \boxtimes \mathcal{G}).$$

Since k is a field, both $\mathcal{F}$ and $\mathcal{G}$ are automatically flat over Spec k; therefore $\mathcal{F} \boxtimes \mathcal{G}$ stays coherent. On the other hand, Δ_Y, and hence its base-change $\Delta_{Y_3/Y_2 \times_k Y_2}$

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[2] We thank Xingzhu Fang for sharing this observation.

has finite tor-dimension, so $\Delta^*_{Y_3/Y_2\times_k Y_2}(\mathcal{F}\boxtimes\mathcal{G})$ is still coherent. Finally, since p_{13} is a base-change of f, it is again proper and of finite cohomological dimension. Hence pushforward along p_{13} preserves coherence.

It is easy to see that the convolution product is associative by writing out the corresponding convolution diagrams with multiple factors. The Grothendieck group $K_0(Y_2) := \pi_0(K(Y_2))$ therefore acquires the structure of an associative algebra via $*$. We remark that taking the derived fiber product $Y \times_X Y$ is important for getting the correct formula for convolution, but as a group, $K_0(Y_2)$ agrees with $K_0((Y_2)^{\mathrm{cl}})$, the Grothendieck group of bounded coherent complexes on the underlying classical stack of Y_2, see [Kha22, Corollary 3.4].

Let us observe that the so-constructed associative algebra has the following properties. To simplify notation, we will write Δ for $\Delta_{Y/X}$ below.

Lemma 1.3. *For any $\mathcal{F} \in \mathcal{D}(Y_2)$ and $\mathcal{G} \in \mathcal{D}(Y)$, we have*

$$\mathcal{F} * (\Delta_*\mathcal{G}) \simeq \mathcal{F} \otimes p_2^*\mathcal{G}$$

$$(\Delta_*\mathcal{G}) * \mathcal{F} \simeq p_1^*\mathcal{G} \otimes \mathcal{F}.$$

Proof. For the first equation, use proper base-change along the cartesian diagram

$$\begin{tikzcd} Y_2 \arrow[r, "\mathrm{id}\times\Delta"] \arrow[d, "p_2"] & Y_3 \arrow[d, "p_{23}"] \\ Y \arrow[r, "\Delta"] & Y_2. \end{tikzcd}$$

For the second, use a similar diagram, with p_2 replaced by p_1, p_{23} replaced by p_{12}, and the top arrow replaced by $\Delta \times \mathrm{id}$. □

This immediately implies the following two corollaries.

Corollary 1.4. *The convolution product on Y_2 restricts to the usual tensor product for complexes supported on the diagonal. Namely, for $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{D}(Y)$, we have*

$$(\Delta_*\mathcal{F}) * (\Delta_*\mathcal{G}) \simeq \Delta_*(\mathcal{F} \otimes \mathcal{G}).$$

In particular, the natural map

$$\Delta_* : K_0(Y) \to K_0(Y_2)$$

maps $(K_0(Y), \otimes)$ to a commutative subalgebra of $(K_0(Y_2), *)$.

Corollary 1.5. *For any $\mathcal{F}$ in $\mathcal{D}(Y_2)$ and $\mathcal{G}$ in $\mathcal{D}(X)$, we have*

$$\mathcal{F} * (\Delta_* f^*\mathcal{G}) \simeq (\Delta_* f^*\mathcal{G}) * \mathcal{F}.$$

In particular, $\Delta_* f^*$ maps $(K_0(X), \otimes)$ to a central subalgebra of $(K_0(Y_2), *)$.

The above setup also provides us a natural representation of the associative algebra $K_0(Y_2)$, provided that Y has some nice properties. Namely, given $\mathcal{K} \in \mathcal{D}(Y_2)$, it defines a functor $\mathrm{Perf}(Y) \to \mathcal{D}(Y)$, sending $\mathcal{F}$ to $p_{1,*}(p_2^*\mathcal{F} \otimes \mathcal{K})$. If we assume Y has the (finite) resolution property that every $\mathcal{F} \in \mathcal{D}(Y)$ has a finite resolution by vector bundles, in which case $\mathcal{D}(Y) \simeq \mathrm{Perf}(Y)$, then the K-theory (for perfect complexes) and the G-theory on Y agree. In this case, we obtain an action

$$K_0(Y_2) \times K_0(Y) \to K_0(Y).$$

It is straightforward to check that when restricted to the subalgebra

$$(K_0(Y), \otimes) \xrightarrow{\Delta_*} (K_0(Y_2), *),$$

this action agrees with the action arising from the usual tensor product on Y. In particular, this action is linear over $K_0(X)$ (via $\Delta_* f^*$), and we obtain a homomorphism from $K_0(Y_2)$ to a matrix algebra

$$K_0(Y_2) \to \mathrm{End}_{K_0(X)}(K_0(Y)).$$

This representation is not an isomorphism in general, and one might think of the failure of it to be an isomorphism as measuring some interesting geometry of the map f. The algebra structure on $K_0(Y_2)$ is in this sense an intrinsic invariant of the map f.

1.6. **Some results in K-theory.** Before we discuss examples of this construction. Let us document some results in K-theory that we will need later.

Theorem 1.7 (Excision)**.** *Let $i : Z \to X$ be a closed immersion of noetherian (derived) algebraic stacks with complementary open immersion $j : U \to X$. Then there is a canonical exact triangle of spectra*

$$K(\mathcal{D}(Z)) \xrightarrow{i_*} K(\mathcal{D}(X)) \xrightarrow{j^*} K(\mathcal{D}(U)).$$

Proof. This is [Kha22, Theorem 3.9]. We provide a sketch of the proof: the category $\mathcal{D}(X)$ (respectively for Z and U) has a bounded t-structure with a noetherian heart $\mathrm{Coh}(X)$, and the K-theory for $\mathcal{D}(X)$ and $\mathrm{Coh}(X)$ are the same by [Bar15, Theorem 6.1]. Moreover, one can show that the abelian category $\mathrm{Coh}(U)$ is a quotient of $\mathrm{Coh}(X)$ by the Serre subcategory $i_* \mathrm{Coh}(Z)$. Therefore [Qui73, § 5, Theorem 5] applies. □

Theorem 1.8 (Homotopy invariance)**.** *Let X be a noetherian (derived) algebraic stack and $\pi : V \to X$ is the affine bundle associated to a torsor under some vector bundle on X. Then pullback along π induces an isomorphism*

$$\pi^* : K_0(X) \simeq K_0(V).$$

Moreover, if V is the trivial torsor, then pullback along the zero section provides an inverse to π^*.

Proof. Use excision and the projective bundle formula, see [Kha22, Theorem 3.17]. ☐

Remark 1.9. This is also called the "Thom isomorphism" in literature. For the K-theory of perfect complexes, homotopy invariance would need regularity assumption on X.

1.10. **Examples.** Let us examine this construction in two examples.

Example 1.11. Let k be $\mathbb{C}$ and denote by $*$ its spectrum. Take $f : Y \to X$ to be the map $*/B \to */G$ induced by the inclusion $B \to G$.

The map f is proper and has finite cohomological dimension, since its fibers are isomorphic to the finite flag variety G/B. The diagonal map $Y \to Y \times_k Y$ is smooth, and in particular has finite tor-amplitude. Moreover, Y satisfies the finite resolution property, since it is projective over $*/G$, while G is linearly reductive, see [Kha22, Proposition 1.35, Example 1.37]. Hence we may apply the above construction in this setting to obtain an associative algebra with a commutative subalgebra, and a central subalgebra in it

$$K_0(Y_2) = K_0(B\backslash G/B) \supset K_0(*/B) \supset K_0(*/G) = R(G).$$

Note that by filtering the unipotent radical U and successively using the homotopy invariance of G-theory, one can show that pullback along the natural map $*/T \to */B$ induces an isomorphism $K_0(*/B) \simeq K_0(*/T) = R(T)$.

In fact, in this case, $R(G)$ agrees with the center and $R(T)$ is a maximal commutative subalgebra. Moreover, the natural representation

$$K_0(Y_2) \to \mathrm{End}_{R(G)}(R(T)) \tag{1}$$

is an isomorphism, see [CG97, Proposition 6.1.19].

Example 1.12. Let $\mathcal{B} = G/B$ over $k = \mathbb{C}$ be the full flag variety. Let $\widetilde{\mathcal{N}} := T^*\mathcal{B}$ be the cotangent bundle of $\mathcal{B}$. It is called the Springer variety. Since $\mathcal{B}$ has a G-action by left multiplication, $\widetilde{\mathcal{N}}$ comes with a moment map, with target being the dual of the lie algebra of G

$$\mu : \widetilde{\mathcal{N}} \to \mathfrak{g}^*.$$

This map is proper and G-equivariant, for the adjoint G-action on $\mathfrak{g}^*$. It factors through the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$, and provides a resolution of singularity of the latter (called the "Springer resolution").

There is, moreover, a natural $\mathbf{G}_m$-action on $\widetilde{\mathcal{N}}$ by dilation in the fiber direction, when thinking of $\widetilde{\mathcal{N}}$ as the cotangent bundle of $\mathcal{B}$. On the other hand, $\mathbf{G}_m$ acts on $\mathfrak{g}^*$ by rescaling it as a vector space. The map μ is equivariant for this action as well. Combined with the G-action, we obtain $\mathbb{G} = G \times \mathbf{G}_m$-actions on both sides of μ, and μ is equivariant for these actions.

We define $\mathbf{St}$ to be the (derived) fiber product $\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$. It inherits a diagonal $\mathbb{G}$-action from the action on the two factors. By derived invariance of G-theory [Kha22, Corollary 3.4], the derived structure of $\mathbf{St}$ will not matter for the purpose of identifying $K(\mathbf{St})$ as a group, so most of the time we will ignore it in this talk.

Now we take $f : Y \to X$ to be

$$\mu : \widetilde{\mathcal{N}}/\mathbb{G} \to \mathcal{N}/\mathbb{G},$$

where for the purpose of identifying the algebra $K_0(\mathbf{St}/\mathbb{G})$ with the correct Hecke algebra, we change the $\mathbf{G}_m$-action in this quotient to be the inverse of the dilation action mentioned above. We can again verify that f is proper and of finite cohomological dimension (since it is representable). The diagonal map $Y \to Y \times_k Y$ is quasi-smooth, since $\widetilde{\mathcal{N}}$ is smooth over k, see [Sta26, Tag 0FDP], and hence it has finite tor-amplitude by [Kha22, Lemma 1.15]. Moreover, $\widetilde{\mathcal{N}}/\mathbb{G}$ again has the finite resolution property by [Kha22, Proposition 1.35], since $\widetilde{\mathcal{N}}$ is quasi-projective and G is linearly reductive.

We may therefore specialize the constructions in § 1.1 in this case to obtain an associative algebra, with a commutative and a central subalgebra

$$K_0(\mathbf{St}/\mathbb{G}) \hookleftarrow K_0(\widetilde{\mathcal{N}}/\mathbb{G}) \simeq R(\mathbb{T}) \hookleftarrow K_0(*/\mathbb{G}) = R(\mathbb{G}).$$

Here for the middle term, we have use the homotopy invariance of G-theory, so that we have

$$K_0(\widetilde{\mathcal{N}}/\mathbb{G}) \simeq K_0(\mathcal{B}/\mathbb{G}) \simeq K_0(*/\mathbb{B}) \simeq R(\mathbb{T}). \tag{2}$$

Moreover, the injectivity of the second arrow is clear, while that of the first arrow will be justified later by Proposition 2.6.

We also obtain a representation

$$\rho_g : K_0(\mathbf{St}/\mathbb{G}) \to \mathrm{End}_{R(\mathbb{G})}(K_0(\widetilde{\mathcal{N}}/\mathbb{G})), \tag{3}$$

where the subscript g stands for "geometric".

Remark 1.13. The two examples are in fact closely related to each other. Namely, the Springer and Steinberg varieties have the following interpretation as incidence varieties

$$\widetilde{\mathcal{N}} = \{(x, n) \mid n \in \mathfrak{b}_x^{\perp}\} \subset \mathcal{B} \times \mathcal{N} \tag{4}$$

$$\mathbf{St} = \{(x_1, x_2, n) \mid n \in \mathfrak{b}_{x_1}^{\perp} \cap \mathfrak{b}_{x_2}^{\perp}\} \subset \mathcal{B} \times \mathcal{B} \times \mathcal{N}, \tag{5}$$

where $\mathfrak{b}_x$ denotes the Lie algebra of the Borel subgroup corresponding to the point x, and $\mathfrak{b}_x^{\perp} \subset \mathfrak{g}^*$ is its annihilator under the evaluation pairing.

From Equation (5), we see there that $\mathbf{St}$ sits inside $\mathcal{B} \times \widetilde{\mathcal{N}}$ as a (derived) closed subscheme. Let us denote the (derived) closed immersion by $\iota : \mathbf{St} \to \mathcal{B} \times \widetilde{\mathcal{N}}$.

Regarding $\widetilde{\mathcal{N}}$ as the cotangent bundle of $\mathcal{B}$, it has a zero section $\eta : \mathcal{B} \to \widetilde{\mathcal{N}}$. Then the following diagram is commutative

$$\begin{tikzcd} K_0(\mathbf{St}/\mathbb{G}) \arrow[r, "\rho_g"] \arrow[d, "(\mathrm{id}\times\eta)^*\iota_*"] & \mathrm{End}_{R(\mathbb{G})}(K_0(\widetilde{\mathcal{N}}/\mathbb{G})) \arrow[d, "\eta^*", "\sim"'] \\ K_0((\mathcal{B}\times\mathcal{B})/\mathbb{G}) \arrow[r, "\sim"] & \mathrm{End}_{R(\mathbb{G})}(K_0(\mathcal{B}/\mathbb{G})), \end{tikzcd} \tag{6}$$

where in the bottom row, $\mathbb{G}$ acts diagonally on $\mathcal{B}\times\mathcal{B}$ (the $\mathbf{G}_m$ part acts trivially), and the horizontal arrow comes from the isomorphism (1) in Example 1.11, using the relation $(\mathcal{B}\times\mathcal{B})/G \simeq \mathcal{B}\backslash G/\mathcal{B}$. The right vertical is an isomorphism given by the homotopy invariance.

The goal of the rest of the talk is to study the structure of $K_0(\mathbf{St}/\mathbb{G})$ using Diagram (6).

2. Geometry of the Steinberg variety

We now study the geometry of the Steinberg variety further via the natural $\mathbb{G}$-equivariant map $\pi : \mathbf{St} \to \mathcal{B}\times\mathcal{B}$ from Equation (5). In this section, we will treat $\mathbf{St}$ as its underlying classical scheme when discussing its geometric properties. This does not affect the structure of $K_0(\mathbf{St}/\mathbb{G})$ as a group, though the convolution product on it a priori needs the derived structure.

2.1. **Stratification.** First recall that $\mathcal{B}\times\mathcal{B}$ has a stratification into its G-orbits for the diagonal G-action, and it is indexed by the finite Weyl group W, i.e. we have

$$\mathcal{B}\times\mathcal{B} = \coprod_{w\in W} Y_w.$$

Each G-orbit Y_w is an affine bundle over $\mathcal{B}$. More precisely,

$$Y_w = \{(x_1, x_2) \mid B_{x_1}, B_{x_2} \text{ are in relative position } w\} \subset \mathcal{B}\times\mathcal{B},$$

it maps to $\mathcal{B}$ via projection to the second factor. Then the fiber over a point $x_2 \in \mathcal{B}$ is precisely the Schubert cell $\mathcal{B}_w \subset \mathcal{B} = G/B_{x_2}$, which is an affine space. Moreover, the stratum labeled by e, the neutral element in W, is the diagonal copy of $\mathcal{B}$.

Hence by pulling back along π, we obtain a $\mathbb{G}$-equivariant stratification of the Steinberg variety

$$\mathbf{St} = \coprod_{w\in W} Z_w, \tag{7}$$

with $Z_e = \Delta(\widetilde{\mathcal{N}})$. We also define $Z_{\leq w}$ (respectively $Z_{<w}$) to be the preimage under π of $\overline{Y}_w = \coprod_{w'\leq w} Y_w$ (respectively $\coprod_{w'<w} Y_w$), where the order relation is for the Bruhat order. These are closed subschemes of $\mathbf{St}$. Note, however, that

$Z_{\leq w}$ is not the same as the closure of Z_w in Z, which we will denote by $\overline{Z}_w$, as indicated by Example 2.4 below.

The following result is due to Steinberg [Ste76].

Lemma 2.2. *For each $w \in W$, the map π identifies Z_w with the conormal bundle of Y_w in $\mathcal{B} \times \mathcal{B}$.*

Proof. Using the description of $\mathbf{St}$ in Equation (5), we see that for any $(x_1, x_2) \in \mathcal{B} \times \mathcal{B}$, the fiber of π at this point is identified with $\mathfrak{b}_1^{\perp} \cap \mathfrak{b}_2^{\perp}$, where $\mathfrak{b}_i^{\perp} \subset \mathfrak{g}^*$ is the annihilator of the Lie algebra of B_{x_i} under the evaluation pairing, for $i = 1, 2$. This is the kernel of the natural map

$$\mathfrak{b}_1^{\perp} \oplus \mathfrak{b}_2^{\perp} \to \mathfrak{b}_1^{\perp} + \mathfrak{b}_2^{\perp},$$

which identifies with the map

$$T^*(\mathcal{B} \times \mathcal{B})|_{(x_1,x_2)} \to T^*\mathrm{Orb}(x_1, x_2)|_{(x_1,x_2)}.$$

Hence the kernel is precisely the fiber of the conormal bundle to $\mathrm{Orb}(x_1, x_2)$ at (x_1, x_2). $\square$

This in particular means that all Z_w's have the same dimension, leading to the following corollary.

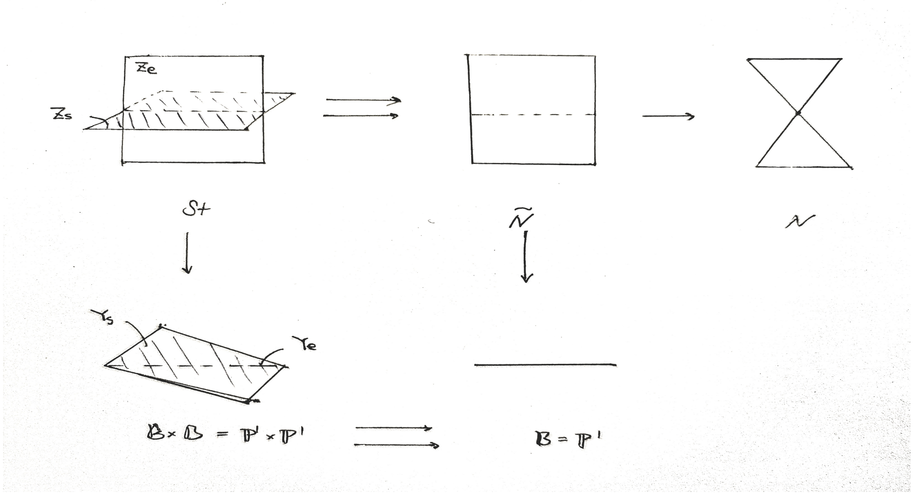
Corollary 2.3. *The irreducible components of* $\mathbf{St}$ *are parametrized by elements of W and are given by the closures of the strata Z_w.*

Example 2.4. Take $G = \mathrm{SL}_2$ over $\mathbb{C}$ and the standard Borel pair. If we represent elements in $\mathfrak{g}^*$ by traceless two-by-two matrices, then the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$ is the subvariety cut out by the condition that the determinant is zero. This is therefore an affine cone isomorphic to $V(x^2 - yz) \subset \mathbb{A}^3_{\mathbb{C}}$. It has du Val singularity of type A_1 at the origin, and the Springer variety $\widetilde{\mathcal{N}}$ is the blowup of $\mathcal{N}$ at the origin, which is also the total space of $\mathcal{O}(\text{-}2)$ on $\mathbb{P}^1 = \mathcal{B}$.

There are two G-orbits on $\mathcal{B} \times \mathcal{B} = \mathbb{P}^1 \times \mathbb{P}^1$,

$$Y_e = \Delta(\mathbb{P}^1),\ Y_s = (\mathbb{P}^1 \times \mathbb{P}^1)\backslash\Delta,$$

for e being the neutral element in W and s being the simple reflection. Hence the Steinberg variety $\mathbf{St}$ has two irreducible components, which are the closures of the two strata Z_e and Z_s. The map π is an isomorphism on $\overline{Z}_s$, and identifies $Z_e \simeq \widetilde{\mathcal{N}}$ with the conormal bundle of $Y_e \subset \mathbb{P}^1 \times \mathbb{P}^1$. In particular, $Z_{\leq s} = \mathbf{St}$ is not the same as $\overline{Z}_s$. Here is a picture of these objects.



2.5. **Consequences.** The geometric structure of $\mathbf{St}$ leads to some consequences on the structure of the algebra $K_0(\mathbf{St}/\mathbb{G})$ and we discuss this in this subsection. We fix an isomorphism $R(\mathbb{T}) \simeq K_0(\widetilde{\mathcal{N}}/\mathbb{G})$ as in Equation (2) and thus think of $R(\mathbb{T})$ as a subalgebra of $K_0(\mathbf{St}/\mathbb{G})$ via pushforward along the diagonal embedding $\Delta : \widetilde{\mathcal{N}} \to \mathbf{St}$.

Proposition 2.6. *As an $R(\mathbb{T})$-module, $K_0(\mathbf{St}/\mathbb{G})$ has a filtration indexed by W, whose graded pieces are given by $K_0(Z_w/\mathbb{G})$. In particular, $K_0(\mathbf{St}/\mathbb{G})$ is free of rank $|W|$ over $R(\mathbb{T})$.*

Proof. Using Lemma 1.3, we see that the natural maps $K_0(Z_{\leq w}/\mathbb{G}) \to K_0(\mathbf{St}/\mathbb{G})$ are linear over $R(\mathbb{T})$, and hence their images form a filtration on $K_0(\mathbf{St}/\mathbb{G})$ indexed by W. For each $w \in W$, we have by excision a long exact sequence

$$\to K_1(Z_{\leq w}/\mathbb{G}) \xrightarrow{\alpha} K_1(Z_w/\mathbb{G}) \to K_0(Z_{<w}/\mathbb{G}) \to K_0(Z_{\leq w}/\mathbb{G}) \to K_0(Z_w/\mathbb{G}) \to 0.$$

By homotopy invariance, Lemma 2.2 and the fact that Y_w is an affine bundle over $\mathcal{B}$, we have

$$K(Z_w/\mathbb{G}) \simeq K(Y_w/\mathbb{G}) \simeq K(\mathcal{B}/\mathbb{G}).$$

This implies that on the one hand, the map α has a section induced by pullback along the natural projection

$$Z_{\leq w} \to \mathcal{B} \times \mathcal{B} \xrightarrow{p_2} \mathcal{B}.$$

On the other hand, $K_0(Z_w/\mathbb{G}) \simeq K_0(\mathcal{B}/\mathbb{G}) \simeq R(\mathbb{T})$ is free of rank one. One easily verify that all the isomorphisms involved are morphisms between $R(\mathbb{T})$-modules. ☐

Corollary 2.7. *The representation ρ_g in Equation (3) is faithful.*

Proof. Both $K_0(\mathbf{St}/\mathbb{G})$ and $\mathrm{End}_{R(\mathbb{G})}(K_0(\mathcal{B}/\mathbb{G}))$ are free of rank $|W|^2$ as $R(\mathbb{T})$-module. It suffices to show that they are isomorphic over the generic point of $\operatorname{Spec} R(\mathbb{T})$. This follows from the localization theorem of Thomason.

Namely, let $Z_0 \subset \mathbf{St}$ be the fiber over some base point x_0 of the morphism

$$\mathbf{St} \xrightarrow{\pi} \mathcal{B} \times \mathcal{B} \xrightarrow{p_2} \mathcal{B}.$$

Then there is a commutative diagram

$$\begin{tikzcd}
K_0(\mathbf{St}/\mathbb{G}) \simeq K_0(Z_0/\mathbb{B}) \simeq K_0(Z_0/\mathbb{T}) \arrow[r] \arrow[d, "(\mathrm{id}\times\eta)^*\iota_*"] & K_0(Z_0^{\mathbb{T}}/\mathbb{T}) \arrow[d, equal] \\
K_0(\mathcal{B} \times \mathcal{B}/\mathbb{G}) \simeq K_0(\mathcal{B}/\mathbb{B}) \simeq K_0(\mathcal{B}/\mathbb{T}) \arrow[r] & K_0(\mathcal{B}^{\mathbb{T}}/\mathbb{T}),
\end{tikzcd}$$

where we follow the notation in Diagram (6), while $Z_0^{\mathbb{T}}$ and $\mathcal{B}^{\mathbb{T}}$ mean the $\mathbb{T}$-fixed points of Z_0 and $\mathcal{B}$. But by [Tho92, Théorèm 2.1], both horizontal arrows become isomorphisms after localizing at the generic point of $\operatorname{Spec} R(\mathbb{T})$, so the corollary follows. ☐

2.8. **Some explicit elements in $K_0(\mathbf{St}/\mathbb{G})$.** The stratification (7) also allows us to write down some explicit elements in the algebra $K_0(\mathbf{St}/\mathbb{G})$.

We let $S \subset W$ be the set of simple reflections and denote by Y_s, $s \in S$ the corresponding G-orbits in $\mathcal{B} \times \mathcal{B}$. The closure

$$\overline{Y}_s = Y_e \coprod Y_s$$

is smooth and is a $\mathbb{P}^1$-fibration over $\mathcal{B}$ via the projection to the second factor. Let $\omega_{\overline{Y}_s/\mathcal{B}}$ be the relative canonical bundle of this map.

The closure of the corresponding stratum in the Steinberg variety

$$\overline{Z}_s = T^*_{\overline{Y}_s}(\mathcal{B} \times \mathcal{B})$$

is also smooth. We write

$$i_s : \overline{Z}_s \hookrightarrow \mathbf{St}$$

for the closed immersion of this stratum, and

$$\pi_s : \overline{Z}_s \to \overline{Y}_s$$

for the projection map.

For each $s \in S$, we set

$$\mathcal{Q}_s = i_{s,*}\pi_s^*\omega_{\overline{Y}_s/\mathcal{B}}.$$

On the other hand, we have the natural map

$$\widetilde{\mathcal{N}}/\mathbb{G} \to \mathcal{B}/\mathbb{G} \to */\mathbb{B} \to */\mathbb{T}.$$

Pulling back along this composite map induces the isomorphism in Equation (2). For each character λ of $\mathbb{T}$, corresponding to some line bundle on $*/\mathbb{T}$, we denote the pullback of this line bundle to $\widetilde{\mathcal{N}}/\mathbb{G}$ by $\mathcal{O}_\lambda$.

The classes $[\mathcal{Q}_s]$, $[\mathcal{O}_\lambda]$ in $K_0(\mathbf{St}/\mathbb{G})$ will provide a basis of the latter that matches with Bernstein's presentation of the extended affine Hecke algebra for $\check{G}$. This will be the topic of the next section.

3. KAZHDAN–LUSZTIG ISOMORPHISM

Before stating the result of Kazhdan–Lusztig, let us recall some basic facts about the extended affine Hecke algebra $\mathbb{H}_{\check{G}}$. First of all, we have the following result due to Bernstein.

Proposition 3.1 (Bernstein's presentation)**.** *The extended affine Hecke algebra $\mathbb{H}_{\check{G}}$ (defined using the Coexeter generators and braid relations) has a basis $\{e^\lambda \cdot T_w \mid w \in W, \lambda \in X^*(T)\}$ over $\mathbb{Z}[q, q^{-1}]$, such that*

(1) The set $\{T_w\}_{w\in W}$ spans a $\mathbb{Z}[q,q^{-1}]$-subalgebra isomorphic to the Hecke algebra of the Coxeter group (W,S).
(2) The set $\{e^\lambda\}_{\lambda\in X^*(T)}$ spans a $\mathbb{Z}[q,q^{-1}]$-subalgebra $\mathbb{A}$ isomorphic to $R(\mathbb{T})$.
(3) If $s = s_\alpha \in S$ with $\langle \lambda, \check{\alpha}_s\rangle = 0$, then $T_s e^\lambda = e^\lambda T_s$.
(4) If $s = s_\alpha \in S$ with $\langle \lambda, \check{\alpha}_s\rangle = 1$, then $T_s e^{s(\lambda)} T_s = q \cdot e^\lambda$.

Moreover, the subalgebra $R(\mathbb{G})^W \subset R(\mathbb{T}) \simeq \mathbb{A}$ is central.

We will identify $\mathbb{A}$ with $R(\mathbb{T})$ below. Using the elements in $K_0(\mathbf{St}/\mathbb{G})$ introduced in Section 2.8, we can write down a map from a generating set of $\mathbb{H}_{\check{G}}$ to $K_0(\mathbf{St}/\mathbb{G})$

$$\begin{aligned}\Theta : \{T_s\}_{s\in S} \cup R(\mathbb{T}) &\to K_0(\mathbf{St}/\mathbb{G})\\ T_s &\mapsto -([q\mathcal{Q}_s] + [\mathcal{O}_0])\\ e^\lambda &\mapsto [\mathcal{O}_{-\lambda}].\end{aligned}$$

Recall also that $\mathbb{H}_{\check{G}}$ has a spherical module $\mathbb{M}_{\mathrm{sph}}$ (we denote the corresponding representation by $\rho : \mathbb{H}_{\check{G}} \to \mathrm{End}_{R(\mathbb{G})}(\mathbb{M}_{\mathrm{sph}})$), which is generated as a left $\mathbb{H}_{\check{G}}$-module by the element $\mathbf{e} := \Sigma_{w\in W} T_w$. It is free of rank one over $R(\mathbb{T})$. This means in particular that as an $R(\mathbb{T})$-module, $\mathbb{M}_{\mathrm{sph}}$ is isomorphic to $K_0(\widetilde{\mathcal{N}}/\mathbb{G})$. We fix an isomorphism

$$\Phi : \mathbb{M}_{\mathrm{sph}} = R(\mathbb{T}) \cdot \mathbf{e} \xrightarrow{\sim} K_0(\widetilde{\mathcal{N}}/\mathbb{G}),\ e^\lambda \cdot \mathbf{e} \mapsto [\mathcal{O}_{-\lambda}].$$

Now the Kazhdan–Lusztig isomorphism can be stated as follows:

Theorem 3.2 (Kazhdan–Lusztig, Ginzburg)**.** *The map Θ extends to an isomorphism of $\mathbb{Z}[q,q^{-1}]$-algebras*

$$\Theta : \mathbb{H}_{\check{G}} \xrightarrow{\sim} K_0(\mathbf{St}/\mathbb{G}).$$

Moreover, the following diagram is commutative

$$\begin{tikzcd}
\mathbb{H}_{\check{G}} \arrow[r, "\rho"] \arrow[d, "\Theta"] & \operatorname{End}_{R(\mathbb{G})}(\mathbb{M}_{\mathrm{sph}}) \arrow[d, "\overset{\Phi}{\simeq}"] \\
K_0(\mathbf{St}/\mathbb{G}) \arrow[r, hook, "\rho_g"] & \operatorname{End}_{R(\mathbb{G})}(K_0(\widetilde{\mathcal{N}}/\mathbb{G})).
\end{tikzcd}$$

We only sketch the idea of the proof. For details, the audiences are referred to [CG97, Chapter 7.6].

Proof. One first checks by the localization theorem and direct computations (which eventually reduce to the rank one case) that the following diagram commutes

$$\begin{tikzcd}
\{T_s\}_{s\in S} \cup R(\mathbb{T}) \arrow[r, hook] \arrow[dr, "\Theta"] & \mathbb{H}_{\check{G}} \arrow[r, "\rho"] & \operatorname{End}_{R(\mathbb{G})}(\mathbb{M}_{\mathrm{sph}}) \arrow[d, "\Phi"] \\
& K_0(\mathbf{St}/\mathbb{G}) \arrow[r, hook, "\rho_g"] & \operatorname{End}_{R(\mathbb{G})}(K_0(\widetilde{\mathcal{N}}/\mathbb{G})).
\end{tikzcd}$$

Since $\{T_s\}_{s\in S} \cup R(\mathbb{T})$ generates $\mathbb{H}_{\check{G}}$, this means the image of $\mathbb{H}_{\check{G}}$ under $\Phi \circ \rho$ is contained in the image of ρ_g. But since we also know that ρ_g is injective, this lifts to a unique algebra homomorphism

$$\Theta : \mathbb{H}_{\check{G}} \to K_0(\mathbf{St}/\mathbb{G})$$

as desired.

To show that this is an isomorphism, one can define a filtration on $\mathbb{H}_{\check{G}}$ by setting for each $w \in W$

$$\mathbb{H}_{\check{G}}^{\leq w} = \text{sub } R(\mathbb{T})\text{-module of } \mathbb{H}_{\check{G}} \text{ generated by } \{T_y\}_{y\leq w},$$

and one similarly defines $\mathbb{H}_{\check{G}}^{<w}$. The quotient $\mathbb{H}_{\check{G}}^{\leq w}/\mathbb{H}_{\check{G}}^{<w}$ is a free left $R(\mathbb{T})$-module of rank one with generator T_w. Then one is tasked to show that Θ is filtration preserving and induces isomorphisms on graded pieces, for the filtration on $K_0(\mathbf{St}/\mathbb{G})$ introduced in the proof of Proposition 2.6.

For this, for each $w \in W$, fix a reduced expression $w = s_1 \cdots s_r$. One checks by computation, with aid of Demasure resolutions for the orbit closure in $\mathcal{B} \times \mathcal{B}$, that the convolution product $\mathcal{Q}_{s_1} * \cdots * \mathcal{Q}_{s_r}$ is supported on $Z_{\leq w}$. This implies that under Θ, any operator T_y for $y \leq w$ is sent to an element in $K_0(Z_{\leq w}/\mathbb{G})$ as desired. On the other hand, one can compute that the induced map on graded pieces

$$\Theta : \mathbb{H}_{\check{G}}^{\leq w}/\mathbb{H}_{\check{G}}^{<w} \to K_0(Z_w/\mathbb{G})$$

sends T_w to $[\mathcal{O}_{Z_w}]$ up to an explicit unit in $R(\mathbb{T})$. This concludes the proof that Θ is an isomorphism. $\square$

Example 3.3. In the case $G = \mathrm{SL}_2$, $\check{G} = \mathrm{PGL}_2$, $X^*(T) \simeq \mathbb{Z}$, $W = \{e, s\} \simeq \mathbb{Z}_2$ and the Hecke algebra $\mathbb{H}_{\check{G}}$ is the associative $\mathbb{Z}[q, q^{-1}]$-algebra on three generators X, X^{-1}, T, subject to the relations

$$(T+1)(T-q) = 0, \ X \cdot X^{-1} = X^{-1} \cdot X = 1,$$

and

$$TX^{-1} - XT = (1-q)X.$$

The geometry of the Steinberg variety is discussed in Example 2.4. In particular, the map Θ on the generators is given by

$$\begin{aligned} \Theta : X^n &\mapsto [\mathcal{O}_{-n}] = [i_{e*}\pi_e^*\mathcal{O}(-n)] \\ T &\mapsto -([q\mathcal{Q}_s] + [\mathcal{O}_0]), \end{aligned}$$

where

$$\mathcal{Q}_s = i_{s,*}\pi_s^*\omega_{\overline{Y}_s/\mathbb{P}^1} \simeq i_{s,*}\pi_s^*(\mathcal{O} \boxtimes \mathcal{O}(-2)).$$

Using these expressions, one can compute the convolution product and verify that the relations in $\mathbb{H}_{\check{G}}$ are preserved under Θ. See [CG97, Chapter 7.5] for details of this example.

Remark 3.4. Fix $t \in \mathbb{C}^*$ not a root of unity. As a consequence of the Kazhdan–Lusztig isomorphism, one can give a geometric (exhaustive) construction of simple modules of

$$\mathbb{H}_t := \mathbb{H}_{\check{G}} \otimes_{\mathbb{Z}[q,q^{-1}], q \mapsto t} \mathbb{C}.$$

They are parametrized by $G(\mathbb{C})$-conjugacy classes of triples (s, n, χ), where $s \in G(\mathbb{C})$ is a semisimple element, $n \in \mathcal{N}$ such that $sns^{-1} = tn$, and χ is an irreducible representation of the component group of the simultaneous centralizer in $G(\mathbb{C})$ of (s, n). When $t = p$ is some rational prime, $\mathbb{H}_t$ identifies with the Iwahori Hecke algebra for the p-adic group $\check{G}(\mathbb{Q}_p)$. This parametrization is precisely the (refined) local Langlands conjecture for Iwahoric spherical representations. See [CG97, Chapter 8.1] for more details.

References

[Bar15] Clark Barwick. On exact ∞-categories and the theorem of the heart. *Compositio Mathematica*, 151(11):2160–2186, 2015.

[CG97] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry.* Birkhäuser, 1997.

[Kha22] A.A. Khan. K-theory and G-theory of derived algebraic stacks. *Jpn. J. Math.*, 17:1–61, 2022.

[Qui73] Daniel Quillen. Higher algebraic K-theory: I. In H. Bass, editor, *Higher K-Theories*, pages 85–147, Berlin, Heidelberg, 1973. Springer Berlin Heidelberg.

[Sta26] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2026.

[Ste76] R. Steinberg. On the desingularization of the unipotent variety. *Invent. Math.*, X 36:209–224, 1976.

[Tho92] R. W. Thomason. Une formule de Lefschetz en K-théorie équivariante algébrique. *Duke Mathematical Journal*, 68(3):447 – 462, 1992.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, FINE HALL, WASHINGTON ROAD, PRINCETON, NJ, 08544-1000, USA
Email address: `mz9413@princeton.edu`