GAITSGORY'S CENTRAL FUNCTOR II: CENTRALITY

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Fix an algebraically closed field k and a reductive group scheme G over k as well as its Borel $B \subseteq G$. Last time, we have introduced a group scheme $\mathcal{G}$ over $\mathbf{A}^1_k = \operatorname{Spec}(k[\pi])$ along with a map $\mathcal{G} \to G_{\mathbf{A}^1_k}$ that is an isomorphism on $\mathbf{A}^1_k \setminus 0$ and the inclusion $B \subseteq G$ at 0. Namely, for a $k[\pi]$-algebra R, we define

$$\mathcal{G}(R) = G(R) \times_{G(R/\pi)} B(R/\pi).$$

Similar to before, we have defined the loop groups $\mathrm{L}^+\mathcal{G}$ and $\mathrm{L}\mathcal{G}$ by writing

$$\mathrm{L}^+\mathcal{G}(R) = \mathcal{G}(R[[t]]) \text{ and } \mathrm{L}\mathcal{G}(R) = \mathcal{G}(R((t))),$$

for any $k[\pi]$-algebra R, and also the Hecke stack $\mathcal{H}_{\mathcal{G}} = \mathrm{L}^+\mathcal{G}\backslash\mathrm{L}\mathcal{G}/\mathrm{L}^+\mathcal{G}$. They all live over $\mathbf{A}^1_k$, with fibers the same constructions of corresponding fibers of $\mathcal{G}$. We have introduced the nearby cycle functor, and applied it on $\mathcal{H}_{\mathcal{G}} \to \mathbf{A}^1_k$ to obtain Gaitsgory's central functor.

For definiteness, we only treat the étale case in this talk; the complex analytic case can be treated similarly, and we will give remarks on its treatment if necessary. So fix a coefficient ring that is a $\mathbb{Z}_\ell$-algebra where ℓ is invertible in k, which we will not explicitly mention and use.

In this talk, the monodromy action on nearby cycles is irrelevant. Therefore, we work in a slightly different setting: Consider the Henselization $k\{\pi\}$ of $k[\pi]$ at $\pi = 0$, and let V be the absolute integral closure of $k\{\pi\}$. Then V is an absolutely integrally closed rank-1 valuation ring with residue field k and pseudo-uniformizer π. Denote its fraction field by K. For a stack X over V, we use $j\colon X_K \to X$ to denote the inclusion of the generic fiber, and $i\colon X_k \to X$ to denote that of the special fiber. Then the nearby cycle functor in this setting is

$$\psi = i^* j_* \colon \mathsf{D}(X_K) \to \mathsf{D}(X_k).$$

Last time, by taking $X = \mathcal{H}_{\mathcal{G}_V}$, we have introduced *Gaitsgory's central functor*

$$\mathcal{Z} = \psi\colon \mathsf{D}(\mathcal{H}_{G_K}) \to \mathsf{D}(\mathcal{H}_B).$$

This time, we will prove that $\mathcal{Z}$ deserves the word "central". We roughly follow the argument of [Ans+25, §4.2, §4.3], adapting it to the geometric setting.

Remark 0.1. One can think of $\operatorname{Spec}(K) \to \mathbf{A}^1_k \setminus 0$ as the algebro-geometric counterpart of $\exp\colon \mathbf{A}^1_{\mathbb{C}} \to \mathbf{A}^1_{\mathbb{C}} \setminus 0$ in the complex analytic world, or rather its fiber over the punctured overconvergent infinitesimal neighborhood of 0. In this picture, the counterpart of $\operatorname{Spec}(V)$ in the complex analytic world is the topological space $\mathbb{C} \cup \{-\infty\}$ where a neighborhood basis of $-\infty$ is

$$(\{z \in \mathbb{C} \mid \operatorname{Re}(z) < a\} \cup \{-\infty\})_{a\in\mathbb{R}},$$

or rather its "stalk" near the point $-\infty$.

For notational convenience, for any stack or algebra S over $k[\pi]$, we let

- $\mathsf{D}_S = \mathsf{D}^{\mathrm{ULA}}(\mathcal{H}_{\mathcal{G}_S})$ denote the category of universally locally acyclic [HS23, Definition 3.2] or equivalently suave [HM24, Definition 4.4.1] sheaves over S; by [HS23, Theorem 4.1], when S is an absolutely integrally closed valuation ring with generic point η, this is the same as $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H}_{\mathcal{G}_\eta})$; in particular, $\mathsf{D}_V = \mathsf{D}_K = \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H}_{G_K})$;
- $\mathsf{P}_S \subseteq \mathsf{D}_S$ be the full subcategory of relatively flat perverse sheaves over S, cf. [FS24, Proposition VI.7.7].

1. Fusion

We recall fusion in this setting. Let X be a smooth curve over k. For a $k[\pi]$-algebra R and a finite family of R-points $x = (x_\lambda)_{\lambda \in \Lambda} \in X^\Lambda(R)$, we let:

- Γ_x denote the divisor $\sum_{\lambda \in \Lambda} \Gamma_{x_\lambda}$ of $X_R = X \times_k \mathrm{Spec}(R)$;
- O_x denote (the global section of) the completion of $\mathcal{O}_{X_R}$ along Γ_x;
- K_x denote the ring obtained by inverting the ideal of Γ_x in O_x.

Then O_x and K_x are R-algebras and hence $k[\pi]$-algebras, so it makes sense to talk about $\mathcal{G}(\mathrm{O}_x)$ and $\mathcal{G}(\mathrm{K}_x)$, and the functors

$$x \mapsto \mathcal{G}(\mathrm{O}_x),\ x \mapsto \mathcal{G}(\mathrm{K}_x), \text{ and } x \mapsto \mathcal{G}(\mathrm{O}_x)\backslash\mathcal{G}(\mathrm{K}_x)/\mathcal{G}(\mathrm{O}_x)$$

are stacks over $X^\Lambda_{k[\pi]}$, which we denote by

$$\mathrm{L}^+_{X^\Lambda}\mathcal{G},\ \mathrm{L}_{X^\Lambda}\mathcal{G}, \text{ and } \mathcal{H}_{\mathcal{G},X^\Lambda} = \mathrm{L}^+_{X^\Lambda}\mathcal{G}\backslash\mathrm{L}_{X^\Lambda}\mathcal{G}/\mathrm{L}^+_{X^\Lambda}\mathcal{G},$$

respectively. In other words,

- An x-point of $\mathrm{pt}/\mathrm{L}^+_{X^\Lambda}\mathcal{G}$ is a $\mathcal{G}$-bundle on the formal neighborhood of Γ_x in X_R;
- An x-point of $\mathrm{pt}/\mathrm{L}_{X^\Lambda}\mathcal{G}$ is a $\mathcal{G}$-bundle on the punctured formal neighborhood of Γ_x in X_R;
- An x-point of $\mathcal{H}_{\mathcal{G},X^\Lambda}$ is the data of two $\mathcal{G}$-bundles on the formal neighborhood of Γ_x in X_R along with an identification of their restrictions on the punctured formal neighborhood; we often call this a *modification* along Γ_x.

For any totally ordered partition $\varphi\colon \Lambda \to \{1, \dots, n\}$, we denote by $x_m = x|_{\Lambda_m}$ the Λ_m-family obtained by restricting x. In the lecture on fusion, we have defined the following enhancement of convolution:

Definition 1.1 (external convolution)**.** Let $\mathcal{H}_{\mathcal{G},X^\Lambda,\varphi}$ denote the stack over $X^\Lambda_{k[\pi]}$, whose x-point is the following data:

- $\mathcal{G}$-bundles $E_0, E_1, \dots, E_n$ on the formal neighborhood of Γ_x in X_R;
- modifications $E_{m-1} \dashrightarrow E_m$ along Γ_{x_i} for all $m \in \{1, \dots, n\}$.

Then there is an obvious correspondence

$$\mathcal{H}_{\mathcal{G},X^{\Lambda_1}} \times_{k[\pi]} \cdots \times_{k[\pi]} \mathcal{H}_{\mathcal{G},X^{\Lambda_n}} \leftarrow \mathcal{H}_{\mathcal{G},X^\Lambda,\varphi} \to \mathcal{H}_{\mathcal{G},X^\Lambda}$$

over $X^\Lambda_{k[\pi]}$, where the left arrow is taking adjacent bundles and the right one is taking E_0 and E_n. The *n-fold external convolution* is then defined as the functor

$$\mathsf{D}_{X^{\Lambda_1}_{k[\pi]}} \times \cdots \times \mathsf{D}_{X^{\Lambda_n}_{k[\pi]}} \to \mathsf{D}_{X^\Lambda_{k[\pi]}}$$

given by pull-push along the above correspondence.

Remark 1.2. A priori, Definition 1.1 is only associative but not commutative, nor does it commute with convolutions of the individual factors. However, if we restrict it to the open subscheme

$$X^{\Lambda;\Lambda_1,\dots,\Lambda_n} = \{x \in X^\Lambda \mid \Gamma_{x_i} \cap \Gamma_{x_j} = \varnothing \text{ for all } i \neq j\}$$

of X^Λ, it becomes obviously commutative and commutes with convolutions of the individual factors. In the lecture on fusion, we have used this fact to establish symmetric monoidality of the Satake category.

From now on, we fix a point $0 \in X$ and an isomorphism $k\{\pi\} = \mathcal{O}^{\mathrm{h}}_{X,0}$, inducing a map $h\colon \mathrm{Spec}(k\{\pi\}) \to X$. For a finite set Λ and a Λ-family of $k\{\pi\}$-stacks $S = (S_\lambda)_{\lambda\in\Lambda}$, we view each S_λ as over $X_{k[\pi]}$ using h, and let

- $S^\Theta = \prod_{\lambda\in\Theta} S_\lambda$ for $\Theta \subseteq \Lambda$, where the product is taken over k;
- $\mathsf{D}_S = \mathsf{D}^{\mathrm{ULA}}(\mathcal{H}_{\mathcal{G},X^\Lambda} \times_{X^\Lambda_{k[\pi]}} S^\Lambda)$ denote the category of universally locally acyclic or equivalently suave sheaves over S^Λ;
- $\mathsf{P}_S \subseteq \mathsf{D}_S$ be the full subcategory of relatively flat perverse sheaves over S^Λ.

Note that when Λ is a singleton, this coincides with the previous notation.

Remark 1.3. Keep the above notation. For a totally ordered partition $\varphi\colon \Lambda \to \{1,\dots,n\}$, let $S_m = S|_{\Lambda_m}$ denotes the Λ_m-family obtained by restricting S. By base change, Definition 1.1 gives rise to a functor

$$\mathsf{D}_{S_1} \times \cdots \times \mathsf{D}_{S_n} \to \mathsf{D}_S$$

which we still call the *external convolution*. Remark 1.2 still holds here: if the map $S^\Lambda \to X^\Lambda$ factors through the open subscheme $X^{\Lambda;\Lambda_1,\dots,\Lambda_n} \subseteq X^\Lambda$, then the external convolution $\mathsf{D}_{S_1} \times\cdots\times \mathsf{D}_{S_n} \to \mathsf{D}_S$ is commutative and commutes with convolutions of the individual factors.

Remark 1.4 (fusion revisited)**.** Keep the above notation, and recall that we have defined V as the absolute integral closure of $k\{\pi\}$ and K the fraction field of V. Note that $\mathsf{P}_K = \mathsf{Sat}_{G_K} = \mathsf{Sat}_G$ where the second equality comes from the fact that both K and k are algebraically closed. Applying Remark 1.3 with $\Lambda = \{1,\dots,n\}$, $\varphi = \mathrm{id}$, and $S_1 = \cdots = S_n = \mathrm{Spec}(K)$, we get a functor

$$\mathsf{P}_K \times \cdots \times \mathsf{P}_K \to \mathsf{P}_{K,\dots,K},$$

where $\mathsf{P}_{K,\dots,K}$ is the category of sheaves on $\mathcal{H}_{\mathcal{G},X^n} \times_{X^n_{k[\pi]}} \mathrm{Spec}(K)^n$ that are flat perverse and ULA relatively over $\mathrm{Spec}(K)^n$. By [HS23, Theorem 6.8(ii)], restricting away from the partial diagonals of $\mathrm{Spec}(K)^n$ is fully faithful, so by Remark 1.2 we get a symmetric monoidal fusion product enhancing the convolution. It is not hard to show that this fusion coincides with the one constructed in the earlier lecture on fusion by a limiting argument similar to Lemma 3.3 below.

2. Center in higher algebra

We recall the notion of the center in higher algebra.

Definition 2.1 (center)**.** Let $\mathcal{C}$ be a presentably symmetric monoidal category. Let $\mathcal{O}$ be a coherent operad [HA, Definition 3.3.1.9] and let A be an $\mathcal{O}$-algebra in $\mathcal{C}$. Then $\mathsf{Mod}^{\mathcal{O}}_A(\mathcal{C})$ [HA, Definition 3.3.3.8] is a $\mathcal{C}$-linear presentably $\mathcal{O}$-monoidal category with unit A. We define the *center* of A as the endomorphism object

$$\mathrm{Z}(A) := \underline{\mathrm{End}}_{\mathsf{Mod}^{\mathcal{O}}_A(\mathcal{C})}(A) \in \mathcal{C}.$$

$\mathsf{Z}(A)$ is naturally an $(\mathsf{Assoc} \otimes \mathcal{O})$-algebra, where:

- the associative algebra structure comes from the fact that $\mathsf{Z}(A)$ is an endomorphism object;
- the $\mathcal{O}$-algebra structure comes from the fact that A, as the unit of the $\mathcal{O}$-monoidal category $\mathsf{Mod}_A^{\mathcal{O}}(\mathcal{C})$, is both an $\mathcal{O}$-algebra and an $\mathcal{O}$-coalgebra, and the mapping object from an $\mathcal{O}$-coalgebra to an $\mathcal{O}$-algebra is $\mathcal{O}$-monoidal.

Also, there is naturally an $\mathcal{O}$-algebra map

$$\mathsf{Z}(A) = \underline{\mathrm{End}}_{\mathsf{Mod}_A^{\mathcal{O}}(\mathcal{C})}(A) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(1, A) = A,$$

coming from the free-forgetful adjunction, as the free functor takes $1 \in \mathcal{C}$ to A.

Remark 2.2. Definition 2.1 is very general, but we will only use the case where $\mathcal{C} = \mathsf{Cat}$ and $\mathcal{O} = \mathsf{Assoc} = \mathbb{E}_1$, so $\mathsf{Assoc} \otimes \mathcal{O} = \mathbb{E}_1 \otimes \mathbb{E}_1 = \mathbb{E}_2$. In other words, we will consider the center of a monoidal category as an $\mathbb{E}_2$-monoidal category.

Remark 2.3. For a more general version of the center, see [HA, Definition 5.3.1.12]; Definition 2.1 is the case $\mathcal{D} = \mathsf{Comm}$ the commutative operad, with suitable presentability condition, cf. [HA, Corollary 5.3.1.15, Remark 5.3.1.32].

3. CENTRALITY

In this section, we establish properties of the central functor.

Proposition 3.1 (monoidality). *$\mathcal{Z}\colon \mathsf{D}_K \to \mathsf{D}_k$ is naturally monoidal.*

Proof. By [HS23, Theorem 4.1], the functor $j_*\colon \mathsf{D}_K \to \mathsf{D}_V$ is an equivalence. Now it suffices to show that both pullbacks $i^*\colon \mathsf{D}_V \to \mathsf{D}_k$ and $j^*\colon \mathsf{D}_V \to \mathsf{D}_K$ are monoidal, which is obvious. ☐

Theorem 3.2 (centrality). *The functor $\mathcal{Z}\colon \mathsf{P}_K \to \mathsf{D}_k$ naturally factors through $\mathsf{Z}(\mathsf{D}_k) \to \mathsf{D}_k$, and the resulting functor $\mathsf{P}_K \to \mathsf{Z}(\mathsf{D}_k)$ is naturally $\mathbb{E}_2$-monoidal.*

Proof. By Definition 2.1, we want to prove that the action of D_K on D_k given by

$$\mathsf{D}_K \times \mathsf{D}_k \to \mathsf{D}_k$$

$$(E, F) \mapsto \mathcal{Z}(E) \star F$$

naturally preserves the D_k-bimodule structure. For this, it suffices to monoidally enhance the above action, as then left and right multiplication by $(1, F) \mapsto F$ will show that the action commutes with left and right multiplication of D_k on itself.

For this, for $n \in \mathbb{N}$ we form the commutative diagram

$$\begin{tikzcd}
\mathsf{P}_K \times \cdots \times \mathsf{P}_K \times \mathsf{D}_k \arrow[d] & \mathsf{P}_V \times \cdots \times \mathsf{P}_V \times \mathsf{D}_k \arrow[l, "\sim"'] \arrow[r] \arrow[d] & \mathsf{D}_k \times \cdots \times \mathsf{D}_k \times \mathsf{D}_k \arrow[d] \\
\mathsf{D}_{K,\dots,K,k} & \mathsf{D}_{V,\dots,V,k} \arrow[l] \arrow[r] & \mathsf{D}_{k,\dots,k,k}
\end{tikzcd}$$

where

- the horizontal arrows are pullbacks and thus already monoidal;
- vertical arrows are external convolutions as in Remark 1.3;
- the "$\mathsf{P}_K \times \cdots \times \mathsf{P}_K$" means n copies of P_K and similar for everywhere else.

Since k is our base, $\mathsf{D}_{k,\dots,k,k} = \mathsf{D}_k$ and the right vertical arrow is the convolution of D_k. Therefore, the action of D_K on D_k in question is the functor given by going from the upper left corner to the lower right corner. By Lemma 3.3 below, the lower left arrow is fully faithful, so it suffices to monoidally enhance the left vertical arrow, which factorizes as

$$\mathsf{P}_K \times \cdots \times \mathsf{P}_K \times \mathsf{D}_k \to \mathsf{P}_{K,\dots,K} \times \mathsf{D}_k \to \mathsf{D}_{K,\dots,K} \times \mathsf{D}_k \to \mathsf{D}_{K,\dots,K,k}.$$

By Remark 1.4, the first arrow above is monoidal, even symmetric monoidal. The second arrow above is monoidal by definition. The third arrow is monoidal by Remark 1.3 as $\operatorname{Spec}(K)$ and $\operatorname{Spec}(k)$ are disjoint in $\operatorname{Spec}(V)$ and hence in X.

The way we monoidally enhance the action makes it clear that the resulting functor $\mathsf{P}_K \to \mathsf{Z}(\mathsf{D}_k)$ is $\mathbb{E}_2$-monoidal. □

Lemma 3.3. *For any* $n \in \mathbb{N}$ *and any* !*-able stack* X *over* $\operatorname{Spec}(V)^n$*, the functor*

$$(j^n)^* \colon \mathsf{D}^{\mathrm{ULA}}(X/\operatorname{Spec}(V)^n) \to \mathsf{D}^{\mathrm{ULA}}(X_{\operatorname{Spec}(K)^n}/\operatorname{Spec}(K)^n)$$

is fully faithful.

Proof. We want to show $(j^n)_*(j^n)^*P = P$ for $P \in \mathsf{D}^{\mathrm{ULA}}(X/\operatorname{Spec}(V)^n)$. By suave base change [HM24, Remark 4.5.15(ii)] (with $f\colon \operatorname{Spec}(K)^n \to \operatorname{Spec}(V)^n$ and $g\colon X \to \operatorname{Spec}(V)^n$), the lemma reduces to the case $X = \operatorname{Spec}(V)^n$ and $P = 1$, so we only need to show $(j^n)_*1 = 1$ with $j\colon \operatorname{Spec}(K) \to \operatorname{Spec}(V)$. Recall by [HS23, Theorem 4.1] that (j^*, j_*) is an equivalence, so $j_*1 = 1$, and we only need to raise this equation to external tensor powers.

Write $V = \operatorname{colim} V_\alpha$ and $K = \operatorname{colim} K_\alpha$, where each V_α is a 1-dimensional smooth k-algebra and $K_\alpha = V_\alpha[1/\pi]$. Let

- $j_\alpha\colon \operatorname{Spec}(K_\alpha) \to \operatorname{Spec}(V_\alpha)$,
- $p_\alpha\colon \operatorname{Spec}(V) \to \operatorname{Spec}(V_\alpha)$, and
- $q_\alpha\colon \operatorname{Spec}(K) \to \operatorname{Spec}(K_\alpha)$

denote the maps. Then it is standard [Stacks, `0EYM`] that

$$(j^n)_*1 = \operatorname{colim}(p_\alpha^n)^*(j_\alpha^n)_*1,$$

for all $n \in \mathbb{N}$. Therefore, it remains to see that

$$(\operatorname{colim} p_\alpha^* j_{\alpha *}1)^{\boxtimes_k n} = \operatorname{colim}(p_\alpha^n)^*(j_\alpha^n)_*1.$$

Now colim and p_α^* obviously commute with external tensor powers, and in this case so does $j_{\alpha *}$ as both Verdier dual and lower shriek do. □

Remark 3.4. The complex analytic counterpart of Lemma 3.3 is very intuitive: the inclusion of the topological spaces defined in Remark 0.1

$$j^n \colon \mathbb{C}^n \to (\mathbb{C} \cup \{-\infty\})^n$$

is a homotopy equivalence, and thus has a fully faithful $(j^n)^*$ on ULA sheaves.

Remark 3.5. From the proof of Theorem 3.2 we see that, if we can somehow use fusion to enhance D_K to an $\mathbb{E}_2$-monoidal category, then we can prove a version of Theorem 3.2 with the full D_K in place of P_K.

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