GEOMETRIC SATAKE I: THE SATAKE CATEGORY

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Let k be a field and let G be a reductive group over k. Recall the loop groups
$$\mathrm{L}^+G(R) := G(R[[t]]),\ \mathrm{L}G(R) := G(R((t)))$$
and the affine Grassmannian
$$\mathrm{Gr}_G := \mathrm{L}G/\mathrm{L}^+G.$$
This talk aims to define the sheaf (derived) category on the double coset stack $\mathrm{L}^+G\backslash\mathrm{L}G/\mathrm{L}^+G$, or more classically, that of L^+G-equivariant sheaves on Gr_G, as well as its convolution product and perverse t-structure. For brevity, we denote
$$\mathcal{H} = \mathrm{L}^+G\backslash\mathrm{L}G/\mathrm{L}^+G.$$

It will be clear that part of the below can be done analogously with L^+G replaced by an Iwahori subgroup I and $\mathrm{L}^+G\backslash\mathrm{L}G/\mathrm{L}^+G$ by $I\backslash\mathrm{L}G/I$.

Remark 0.1. Sometimes, we will assume that G is split, choose a split torus and Borel $T \subset B \subset G$, and denote by:

- U the unipotent radical of B;
- $\mathrm{X}_*(T)_+$ the poset of dominant cocharacters with partial order generated by $\mu \leq \mu + \alpha^\vee$ for simple coroots $\alpha^\vee$;
- $2\rho \in \mathrm{X}^*(T)_+$ the sum of all the positive roots;
- $2\rho^\vee \in \mathrm{X}_*(T)_+$ the sum of all the positive coroots.

However, all results here remain valid for general G by étale descent.

1. Sheaf categories for stacks

We want to define the derived category of étale (or complex-analytic when $k = \mathbb{C}$) sheaves on $\mathcal{H} = \mathrm{L}^+G\backslash\mathrm{L}G/\mathrm{L}^+G$. In fact, we can do it for general stacks.

Definition 1.1 (stack)**.** Let Ring_k denote the category of countably presented k-algebras. By *big étale stacks* over k, we mean big étale sheaves of animas over k, i.e. functors $\mathsf{Ring}_k \to \mathsf{Ani}$ satisfying étale descent. We will simply call them *stacks* (over k) below and denote by Stack_k the category of them. Stack_k is a topos.

Remark 1.2. Here we are really using static, i.e. non-animated k-algebras, since we are studying étale sheaves which are insensitive to universal homeomorphisms. The coefficient ring Λ below may also be animated, except for the discussion around the perverse fiber functor.

Definition 1.3 (evaluation of sheaves on stacks)**.** For a presentable category $\mathcal{C}$ and a big étale sheaf $F\colon \mathsf{Ring}_k \to \mathcal{C}$, we right Kan extend along the Yoneda embedding $\mathrm{Spec}\colon \mathsf{Ring}_k^{\mathrm{op}} \to \mathsf{Stack}_k$ and still denote the resulting functor by $F\colon \mathsf{Stack}_k^{\mathrm{op}} \to \mathcal{C}$. In other words, for $X \in \mathsf{Stack}_k$,
$$F(X) := \lim_{\mathrm{Spec}(R)\to X} F(R).$$

By abstract nonsense, if $X = \operatorname{colim}_{i\in I} \operatorname{Spec}(R_i)$, then $F(X) = \lim_{i\in I} F(R_i)$.

Now let Λ be a ring and ℓ a prime invertible in k, and assume one of the following:

(1) $k = \mathbb{R}$ or $k = \mathbb{C}$; in this case, let $\mathsf{D}(-;\Lambda)\colon \mathsf{Ring}_{\mathbb{C}} \to \mathsf{Pr}^{\mathrm{L}}$ denote the category of complex-analytic $\mathsf{D}(\Lambda)$-sheaves with $*$-pullback functoriality.

(2) Λ is ℓ-power torsion; in this case, let $\mathsf{D}(-;\Lambda)\colon \mathsf{Ring}_k \to \mathsf{Pr}^{\mathrm{L}}$ denote the category of small étale $\mathsf{D}(\Lambda)$-sheaves with $*$-pullback functoriality.

(3) Λ is a light condensed $\mathbb{Z}_\ell$-algebra; in this case, view Λ as a pro-étale sheaf of rings on each $R \in \mathsf{Ring}_k$ and let $\mathsf{D}(-;\Lambda)\colon \mathsf{Ring}_k \to \mathsf{Pr}^{\mathrm{L}}$ be the category of Λ-complexes with $*$-pullback functoriality. People very familiar with condensed mathematics should actually assume that Λ is nuclear and use the nuclear variant as in [Man22], while people unfamiliar with condensed mathematics can use whatever adic formalism that they like.

In any of the above situations, Definition 1.3 extends the formalism to a functor $\mathsf{D}(-;\Lambda)\colon \mathsf{Stack}_k^{\mathrm{op}} \to \mathsf{Pr}^{\mathrm{L}}$. In other words, for any stack X we have a presentable stable category $\mathsf{D}(X;\Lambda)$ and for any map $f\colon Y \to X$ we have a pair of adjoint functors $f^*\colon \mathsf{D}(X;\Lambda) \rightleftarrows \mathsf{D}(Y;\Lambda) : f_*$. In fact, using [HM24, §3.4] we can also extend the !-functors to a fairly large class of stack maps, but we will not need it here, because essentially we are only going to !-pushforward along proper maps.

Recall from the previous talk that Gr_G is an ind-proper ind-scheme with an L^+G-equivariant stratification indexed by $\mathrm{X}_*(T)_+$, and the stratum Gr_G^μ corresponding to $\mu \in \mathrm{X}_*(T)_+$ is smooth of dimension $d_\mu := \langle 2\rho, \mu\rangle$. Therefore,

$$\mathrm{Gr}_G = \operatorname*{colim}_{\mu\in \mathrm{X}_*(T)_+} \mathrm{Gr}_G^{\leq\mu}$$

is a filtered colimit along closed immersions. Taking quotient by L^+G, we get a similar stratification of $\mathcal{H}$, whose strata we denote by $\mathcal{H}^\mu$. We have a similar filtered colimit along closed immersions

$$\mathcal{H} = \operatorname*{colim}_{\mu\in \mathrm{X}_*(T)_+} \mathcal{H}^{\leq\mu}.$$

Remark 1.4. As recalled in Proposition 2.1 below, for $R \in \mathsf{Ring}_k$, an R-point of $\mathcal{H}$ is the data of two G-bundles on $R[[t]]$ along with an identification of their restrictions on $R((t))$. From this viewpoint, $\mathcal{H}^\mu$ can be intuitively thought of as the locus where the two G-bundles have "difference" $\mu \in \mathrm{X}_*(T)_+$.

More precisely, for $R \in \mathsf{Ring}_k$, an R-point of $\mathcal{H}^\mu$ is the data of two G-bundles E_0 and E_1 on $R[[t]]$ with an identification $e\colon E_0|_{R((t))} = E_1|_{R((t))}$, such that étale locally on R, there are trivializations of E_0 and E_1, with respect to which the matrix of e is $t^\mu \in T(R((t))) \subset G(R((t)))$.

$\mathsf{D}(\mathcal{H};\Lambda)$ is too large for later purposes, so we specify a full subcategory.

Definition 1.5 (perfect constructible sheaves)**.** Let $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H};\Lambda) \subset \mathsf{D}(\mathcal{H};\Lambda)$ be the full subcategory of objects that are

- supported on the closed substack $\mathcal{H}^{\leq\mu}$ for some $\mu \in \mathrm{X}_*(T)_+$;
- locally constant with values perfect Λ-complexes on each $\mathcal{H}^\mu$.

Remark 1.6. Since L^+G acts transitively on each Gr_G^μ, local constancy is actually automatic in situations (1) and (2) above. However, in situation (3) above local constancy is not automatic (as a consequence of not using the nuclear formalism), so I still choose to include it in the definition.

Remark 1.7. It is not hard to see that the stratification of $\mathcal{H}$ is independent of the split torus and Borel, so for general G one can define $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H};\Lambda)$ by étale descent. Similar remarks work below and we will omit them.

2. Convolution and fiber functor

We want to define convolution and fiber functor on $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H})$, and more generally on $\mathsf{D}(H\backslash G/H)$ for any group homomorphism $H \to G$. For this, denote $X = \mathrm{pt}/G$ and $Y = \mathrm{pt}/H$. A group homomorphism $H \to G$ gives a stack map $Y \to X$. Recall:

Proposition 2.1. *$H\backslash G/H = Y \times_X Y$. In other words, a point in $H\backslash G/H$ is the data of two H-torsors along with an identification of their G-inductions.*

Proof. We prove a more general statement: for an object Z with a G-biaction, i.e. a left G-action and a right G-action that commute, the square

$$\begin{tikzcd} H\backslash Z/H \arrow[r] \arrow[d] & H\backslash Z/G \arrow[d] \\ G\backslash Z/H \arrow[r] & G\backslash Z/G \end{tikzcd}$$

is a pullback. Note that the square can be obtained by pulling back the same square with $Z = \mathrm{pt}$ along the classifying map $G\backslash Z/G \to G\backslash \mathrm{pt}/G$, so it suffices to treat the case $Z = \mathrm{pt}$, which is obvious. $\square$

We first review the general construction for convolution on $\mathsf{D}(Y \times_X Y)$. By six-functor nonsense, it suffices to give an algebra structure to $Y \times_X Y$ viewed as an object in the correspondence category, and then use pull-push.

Definition 2.2 (convolution)**.** We give $Y \times_X Y \in \mathsf{Corr}$ the following algebra structure: for $n \in \mathbb{N}$, the n-fold product is defined as the correspondence

$$(Y \times_X Y)^n \leftarrow (Y \times_X Y) \times_Y \cdots \times_Y (Y \times_X Y) = Y^{\times_X \{0,\ldots,n\}} \to Y \times_X Y$$

so the n-fold *convolution* on $\mathsf{D}(Y \times_X Y)$ is given by the following process:

- $*$-pullback the n sheaves to $\mathsf{D}(Y^{\times_X \{0,\ldots,n\}})$ along the projections to factors $\{0,1\}$, $\{1,2\}$, $\ldots$, $\{n-1,n\}$.
- $\otimes$ on $\mathsf{D}(Y^{\times_X \{0,\ldots,n\}})$.
- $!$-pushforward back to $\mathsf{D}(Y \times_X Y)$ along the projection to factors $\{0,n\}$.

In particular, for $A, B \in \mathsf{D}(Y \times_X Y)$, their convolution $A \star B$ is given by

$$\mathsf{D}(Y \times_X Y)^2 \xrightarrow{\mathrm{pr}_{01}^* \otimes \mathrm{pr}_{12}^*} \mathsf{D}(Y \times_X Y \times_X Y) \xrightarrow{(\mathrm{pr}_{02})_!} \mathsf{D}(Y \times_X Y),$$

$$(A, B) \mapsto \mathrm{pr}_{01}^*(A) \otimes \mathrm{pr}_{12}^*(B) \mapsto (\mathrm{pr}_{02})_!(\mathrm{pr}_{01}^*(A) \otimes \mathrm{pr}_{12}^*(B)).$$

Associativity is clear by proper base change and projection formula.

Remark 2.3 (convolution dual)**.** By six-functor nonsense, if $A \in \mathsf{D}(Y \times_X Y)$ is suave over X, i.e. admits a Verdier dual $\mathbb{D}A$ relative to X, then it is convolution-dualizable with dual $\mathrm{sw}^*\mathbb{D}A$, where $\mathrm{sw}\colon Y \times_X Y \to Y \times_X Y$ denotes the automorphism of $Y \times_X Y$ switching the two factors. In particular, this applies to the case $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H};\Lambda)$ below, showing that it is convolution-rigid.

Now assume that the spaces Y and X and the map $Y \to X$ are all pointed, which holds true in the situation $X = \mathrm{pt}/G$ and $Y = \mathrm{pt}/H$.

Definition 2.4 (fiber functor)**.** Consider the correspondence

$$Y \times_X Y \leftarrow \mathrm{pt} \times_X Y \to \mathrm{pt}$$

The *fiber functor* $\mathsf{D}(Y \times_X Y) \to \mathsf{D}(\mathrm{pt})$ is the pull-push along this correspondence.

Remark 2.5. In this generality, the convolution may not be commutative and the fiber functor may not be (symmetric) monoidal, but in the special case of $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H};\Lambda)$ both will hold, as we will see in later talks using fusion product.

Now specialize to the case where $Y = \mathrm{pt}/\mathrm{L}^+G$ and $X = \mathrm{pt}/\mathrm{L}G$, so $Y \times_X Y = \mathcal{H}$ by Proposition 2.1. For $n \in \mathbb{N}$, denote by

$$\mathcal{H}_n := \mathcal{H} \times_Y \cdots \times_Y \mathcal{H} = (Y \times_X Y) \times_Y \cdots \times_Y (Y \times_X Y) = Y^{\times_X \{0,\dots,n\}}$$

the n-fold fiber product of $\mathcal{H}$ over Y, consider the $\mathrm{X}_*(T)^n_+$-stratification of $\mathcal{H}_n$ by

$$\mathcal{H}^\lambda_n := \mathcal{H}^{\lambda_1} \times_Y \cdots \times_Y \mathcal{H}^{\lambda_n}$$

for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathrm{X}_*(T)^n_+$, and analogously to Definition 1.5 define the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H}_n;\Lambda)$ of perfect $\mathrm{X}_*(T)^n_+$-constructible sheaves on $\mathcal{H}_n$. Note that

- $Y \to X$ is represented by the ind-proper ind-scheme Gr_G;
- for any $\mu \in \mathrm{X}_*(T)_+$, both projections $\mathcal{H}^{\le\mu} \subset \mathcal{H} = Y \times_X Y \rightrightarrows Y$ are proper;

so for any $n \in \mathbb{N}$ and $\lambda \in \mathrm{X}_*(T)^n_+$, the projection $\mathcal{H}^\lambda_n \subset \mathcal{H}_n \to \mathcal{H}$ occurred as the right leg of the diagram in Definition 2.2 is proper, and thus the convolution on $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H};\Lambda)$ is well-defined and associative. Similarly, the fiber functor $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H};\Lambda) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathrm{Spec}(k);\Lambda)$ is also well-defined.

3. Perversity

We want to define the notion of perversity on $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H}_n;\Lambda)$ using the stratification of $\mathcal{H}_n$ by $\mathrm{X}_*(T)^n_+$. Below, for $n \in \mathbb{N}$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathrm{X}_*(T)^n_+$, we denote by:

- $i_\lambda \colon \mathcal{H}^\lambda_n \to \mathcal{H}_n$ the locally closed immersion of the stratum $\mathcal{H}^\lambda_n$;
- $i_{\le\lambda} \colon \mathcal{H}^{\le\lambda}_n \to \mathcal{H}_n$ the closed immersion of the closure $\mathcal{H}^{\le\lambda}_n = \overline{\mathcal{H}^\lambda_n}$;
- d_λ the number $d_{\lambda_1} + \cdots + d_{\lambda_n} = \langle 2\rho, \lambda_1 + \cdots + \lambda_n\rangle$.

Definition 3.1 (perversity on $\mathcal{H}_n$)**.** Fix $n \in \mathbb{N}$. An object $A \in \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H}_n;\Lambda)$ is *flat perverse* if for every $\lambda \in \mathrm{X}_*(T)^n_+$, the (cohomological) tor-amplitude of $i^*_\lambda A$ is $\le -d_\lambda$, and that of $i^!_\lambda A$ is $\ge d_\lambda$.

More generally, for $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, an object $A \in \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H}_n;\Lambda)$ is of *perverse tor-amplitude* concentrated on $[a, b]$ if for every $\lambda \in \mathrm{X}_*(T)^n_+$, the tor-amplitude of $i^*_\lambda A$ is $\le b - d_\lambda$, and that of $i^!_\lambda A$ is $\ge a + d_\lambda$.

We call the full subcategory of flat perverse sheaves on $\mathcal{H}$ the *Satake category*, and denote it by $\mathsf{Sat}(G;\Lambda) \subset \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H};\Lambda)$.

Remark 3.2. A more common practice in the literature is to assume that Λ is regular noetherian and define the perverse t-structure in $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H}_n;\Lambda)$ by saying that the connective objects are those with i^*_λ concentrated on $\le -d_\lambda$ while the coconnective objects are those with $i^!_\lambda$ concentrated on $\ge d_\lambda$. I learn the treatment of Definition 3.1 from [FS24, §VI.7.1]. It has the advantage that it needs no assumption on Λ and that the inclusion $\mathsf{Sat}(G;\Lambda) \subset \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H};\Lambda)$ is always (symmetric) monoidal. Under the Satake equivalence, $\mathsf{Sat}(G;\Lambda)$ will correspond to the representation category of the dual group on finite projective Λ-modules.

Remark 3.3 (relative perversity)**.** Since $\mathcal{H}_n$ is highly stacky, it is hard to copy the usual notion of middle perversity to $\mathcal{H}_n$ directly. However, Definition 3.1 does have a close relation to the usual notion via relative perversity:

Since for $\mu \in \mathrm{X}_*(T)_+$, Gr_G^μ is smooth of dimension d_μ, we can see that for $\lambda \in \mathrm{X}_*(T)_+^n$, both projections $\mathcal{H}_n^\lambda \rightrightarrows \mathrm{pt}/\mathrm{L}^+G$ are smooth with dimension d_λ, so an object $A \in \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H}_n;\Lambda)$ is flat perverse if and only if it is relatively flat perverse along either projection $\mathcal{H}_n \to \mathrm{pt}/\mathrm{L}^+G$. Here, by relative perversity along a map, we mean perversity of the $*$-restriction to the fiber (i.e. Gr_G here), for which we are using the usual notion of middle perversity, cf. [HS23].

In particular, for $n = 1$, Definition 3.1 is equivalent to the perversity notion after $*$-pullback along $\mathrm{Gr}_G \to \mathcal{H}$.

Next we want to see that the convolution product preserves the Satake category. Recall that the n-fold convolution product is defined as the composite

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H};\Lambda)^n \xrightarrow{\boxtimes_{\mathrm{pt}/\mathrm{L}^+G}} \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H}_n;\Lambda) \xrightarrow{m_!} \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H};\Lambda)$$

where $m\colon \mathcal{H}_n \to \mathcal{H}$ denotes the projection map, i.e. the right leg of the diagram in Definition 2.2. Since box product preserves the usual middle perversity and by Remark 3.3 the perversity in $\mathcal{H}_n$ is the relative perversity over $\mathrm{pt}/\mathrm{L}^+G$, we see that the first step, i.e. box product over $\mathrm{pt}/\mathrm{L}^+G$, preserves perversity. It remains to see that the second step also preserves perversity. For this we introduce:

Definition 3.4 (semismallness)**.** Let X and Y be k-schemes of finite type with smooth stratifications $(X_\mu)_{\mu\in\mathrm{M}}$ and $(Y_\nu)_{\nu\in\mathrm{N}}$ where M and N are finite posets. A (not necessarity stratified) map $f\colon X \to Y$ is *semismall* if for all $\mu \in \mathrm{M}$, $\nu \in \mathrm{N}$ and $y \in Y_\nu$,

$$2\dim(f^{-1}(y) \cap X_\mu) \leq \dim(X_\mu) - \dim(Y_\nu).$$

Remark 3.5. Despite the apparent symmetry, the roles that M and N play are different in Definition 3.4. N and the stratification on Y are purely auxiliary. One can also state the definition with an unstratified Y and require that there exists such a stratification on it. This will be illustrated by the following proposition.

Proposition 3.6. *Keep the notations in Definition 3.4. Assume that Λ is a field. Let $\mathsf{D}^{\mathrm{b}}_{\mathrm{M\text{-}cons}}(X;\Lambda)$ denote the derived category of* M*-constructible* $\mathsf{Perf}(\Lambda)$*-sheaves on X and $\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(Y;\Lambda)$ denote the derived category of constructible* $\mathsf{Perf}(\Lambda)$*-sheaves on Y. Equip them with the usual middle perverse t-structures. Then*

(1) $f_!$ is right t-exact.

(2) f_* is left t-exact.

In particular, if f is proper, then $f_! = f_*$ is t-exact. For general Λ, the same hold with left/right t-exactness replaced by preserving perverse tor-amplitude $\geq 0/\leq 0$.

Proof. We prove the case of a field. The general case follows either by exactly the same argument or by reduction to the field case.

(1) and (2) imply each other by Verdier duality. We prove (1).

Since connective objects in $\mathsf{D}^{\mathrm{b}}_{\mathrm{M\text{-}cons}}(X;\Lambda)$ are generated under finite colimits and extensions by !-pushforwards of perverse local systems on strata, it suffices to treat the case where X has only one stratum, say of dimension d. In this case, we need to show that $f_!L[d] \in \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(Y;\Lambda)$ is perverse connective for any local system of vector spaces L. Now the semismallness condition becomes

$$2\dim(f^{-1}(y)) \leq d - \dim(Y_\nu)$$

for any $\nu \in \mathrm{N}$ and $y \in Y_\nu$; for any geometric point $\bar{y} \to y$, proper base change gives

$$(f_! L)_{\bar{y}} = \Gamma_{\mathrm{c}}(L|_{f^{-1}(\bar{y})}) \in \mathsf{D}^{[0, 2\dim(f^{-1}(y))]}(\Lambda);$$

shifting by d, this exactly gives that $f_! L[d]$ is concentrated on degrees $\leq -\dim(Y_\nu)$ on Y_ν, i.e. that $f_! L[d]$ is perverse connective. □

Now we come back to the question whether $m_! \colon \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H}_n; \Lambda) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H}; \Lambda)$ preserves perversity. Clearly we only need to treat the cases $n = 0$ and $n = 2$, and the case $n = 0$ is obvious, because $\mathcal{H}_0 \to \mathcal{H}$ is a closed immersion, being the diagonal of the ind-proper ind-schematic map $\mathrm{pt}/\mathrm{L}^+G \to \mathrm{pt}/\mathrm{L}G$. The case $n = 2$ reduces to semismallness of the map $m \colon \mathcal{H}_2 \to \mathcal{H}$ relatively over $\mathrm{pt}/\mathrm{L}^+G$.

Using the moduli description in Remark 1.4, an R-point of $\mathcal{H}_2$ is the data of three G-bundles E_0, E_1, E_2 on $R[[t]]$ with identifications $e_{01} \colon E_0|_{R((t))} = E_1|_{R((t))}$ and $e_{12} \colon E_1|_{R((t))} = E_2|_{R((t))}$, and the map m sends it to the data $(E_0, E_2; e_{12}e_{01})$.

Take $\lambda, \mu, \nu \in \mathrm{X}_*(T)_+$ and consider $(m|_{\mathcal{H}_2^{\lambda,\mu}})^{-1}(L_\nu)$, where L_ν denotes the k-point of two trivial G-bundles connected by $t^\nu \in T(k((t))) \subset G(k((t)))$. Then an R-point of $(m|_{\mathcal{H}_2^{\lambda,\mu}})^{-1}(L_\nu)$ is the above data $(E_0, E_1, E_2; e_{01}, e_{12})$ with E_0 and E_2 trivialized such that $e_{12}e_{01} = t^\nu$, i.e. the data of $(E_1, e_{10}) \in \mathrm{Gr}_G^\lambda$ and $(E_1, e_{12}) \in \mathrm{Gr}_G^\mu$ such that $e_{12} = t^\nu e_{10}$. Therefore, the semismallness in question translates to

Lemma 3.7. *For $\lambda, \mu, \nu \in \mathrm{X}_*(T)_+$,*

$$\dim(\mathrm{Gr}_G^\mu \cap t^\nu \mathrm{Gr}_G^\lambda) \leq \langle \rho, \lambda + \mu - \nu \rangle.$$

Before going into the proof, we first explain what happens for $G = \mathrm{SL}_2$. We choose T to be the diagonal matrices, and identify $\mathrm{X}_*(T)$ with $\mathbb{Z}$ through the map

$$\mathbb{Z} \to \mathrm{Hom}(\mathbf{G}_{\mathrm{m}}, T),\ \lambda \mapsto (t \mapsto \mathrm{diag}(t^\lambda, t^{-\lambda})),$$

so $\rho = 1 \in \mathbb{Z} = \mathrm{X}^*(T)$, as the positive root of SL_2 is 2. Recall the $\mathrm{X}_*(T) = \mathbb{Z}$-stratification of $\mathrm{Gr}_{\mathrm{SL}_2}$ by the semi-infinite orbits

$$S_\kappa(R) = \begin{bmatrix} t^\kappa & t^{\kappa-1}R[t^{-1}] \\ 0 & t^{-\kappa} \end{bmatrix} \mathrm{SL}_2(R[[t]]) \subset \mathrm{Gr}_{\mathrm{SL}_2}(R).$$

Note that $t^{-\nu} S_\kappa = S_{\kappa-\nu}$, and $S_\kappa \cap \mathrm{Gr}_{\mathrm{SL}_2}^{\leq\lambda} \neq \emptyset$ if and only if $|\kappa| \leq \lambda$, in which case

$$(S_\kappa \cap \mathrm{Gr}_{\mathrm{SL}_2}^{\leq\lambda})(R) = \begin{bmatrix} t^\kappa & Rt^{\kappa-1} + \cdots + Rt^{-\lambda} \\ 0 & t^{-\kappa} \end{bmatrix} \mathrm{SL}_2(R[[t]]),$$

so $\dim(S_\kappa \cap \mathrm{Gr}_{\mathrm{SL}_2}^{\leq\lambda}) = \kappa + \lambda$. Therefore,

$$\begin{aligned} \dim(\mathrm{Gr}_G^\mu \cap t^\nu \mathrm{Gr}_G^\lambda) &\leq \max\{\dim(S_\kappa \cap t^\nu \mathrm{Gr}_G^{\leq\lambda}) \mid S_\kappa \cap \mathrm{Gr}_G^{\leq\mu} \neq \emptyset\} \\ &\leq \max\{\lambda + \kappa - \nu \mid \kappa \leq \mu\} = \lambda + \mu - \nu. \end{aligned}$$

Proof. In general, recall the $\mathrm{X}_*(T)$-stratification of Gr_G by the semi-infinite orbits $S_\kappa := \mathrm{L}Ut^\kappa\mathrm{L}^+G \subset \mathrm{Gr}_G$ (or alternatively

$$S_\kappa := \{x \in \mathrm{Gr}_G \mid \lim_{\mathbf{G}_{\mathrm{m}} \ni s \to 0} (2\rho^\vee)(s)x = t^\kappa \mathrm{L}^+G\}),$$

which satisfies:

- $t^{-\nu} S_\kappa = S_{\kappa-\nu}$;
- $S_\kappa \cap \mathrm{Gr}_G^{\leq\lambda} \neq \emptyset$ if and only if every conjugate of κ is $\leq \lambda$, in which case

$$\dim(S_\kappa \cap \mathrm{Gr}_G^{\leq\lambda}) = \langle \rho, \kappa + \lambda \rangle.$$

Therefore,

$$\begin{aligned}\dim(\mathrm{Gr}_G^{\mu}\cap t^{\nu}\mathrm{Gr}_G^{\lambda}) &\leq \max\{\dim(S_\kappa\cap t^{\nu}\mathrm{Gr}_G^{\leq\lambda}) \mid S_\kappa\cap \mathrm{Gr}_G^{\leq\mu}\neq\emptyset\}\\ &\leq \max\{\langle\rho,\lambda+\kappa-\nu\rangle \mid \kappa\leq\mu\} = \langle\rho,\lambda+\mu-\nu\rangle. \qquad\square\end{aligned}$$

Remark 3.8. Alternatively, one can use fusion to prove that convolution preserves perversity. See for example [Zhu16, Proposition 5.4.2].

Remark 3.9 (rigidity)**.** Now that we have seen that convolution preserves flat perversity, Remark 2.3 implies that $\mathsf{Sat}(G;\Lambda)$ is also convolution-rigid.

Finally we define the perverse fiber functor. Unlike the convolution product, the fiber functor in Definition 2.4 is not t-exact. Here we instead take the direct sum of all the cohomology sheaves.

Theorem 3.10 (perverse fiber functor)**.** *For any $A\in\mathsf{Sat}(G;\Lambda)\subset\mathsf{D}^{\mathrm{b}}_{\mathrm{cons}}(\mathcal{H};\Lambda)$, its pull-push along the correspondence $\mathcal{H}\leftarrow\mathrm{Gr}_G\to\mathrm{pt}$ has cohomology sheaves local systems of finite projective Λ-modules. Moreover, the* fiber functor $\mathsf{Sat}(G;\Lambda)\to\mathsf{LocSys}(\mathrm{Spec}(k);\Lambda)$*, given by direct summing all the cohomology sheaves, is exact, faithful, conservative, and reflects kernels and cokernels.*

We only sketch a proof here. It uses hyperbolic localization which will be discussed in later talks. We can assume that the base field k is algebraically closed.

Proof. Here, we use both the $\mathrm{X}_*(T)$-stratification

$$S_\kappa := \{x\in\mathrm{Gr}_G \mid \lim_{\mathbf{G}_{\mathrm{m}}\ni s\to 0}(2\rho^\vee)(s)x = t^\kappa\mathrm{L}^+G\},$$

and the $\mathrm{X}_*(T)^{\mathrm{op}}$-stratification

$$S_\kappa^- := \{x\in\mathrm{Gr}_G \mid \lim_{\mathbf{G}_{\mathrm{m}}\ni s\to \infty}(2\rho^\vee)(s)x = t^\kappa\mathrm{L}^+G\}.$$

For $\kappa\in\mathrm{X}_*(T)$, denote by $j_\kappa\colon S_\kappa\to\mathrm{Gr}_G$ and $j_\kappa^-\colon S_\kappa^-\to\mathrm{Gr}_G$ the locally closed immersions of strata; then hyperbolic localization gives a natural isomorphism

$$\Gamma_{\mathrm{c}}\circ j_\kappa^*\cong\Gamma\circ j_\kappa^{-!}$$

for $\mathbf{G}_{\mathrm{m}}$-equivariant sheaves on Gr_G, in particular for sheaves pulled back from $\mathcal{H}$.

It suffices to prove the stronger statement that for $A\in\mathsf{Sat}(G;\Lambda)$ viewed as an L^+G-equivariant sheave on Gr_G, the perverse fiber functor applied on A equals

$$\bigoplus_{\kappa\in\mathrm{X}_*(T)}\Gamma_{\mathrm{c}}(j_\kappa^*A)[d_\kappa].$$

For $\lambda\in\mathrm{X}_*(T)_+$, by perversity i_λ^*A has tor-amplitude $\leq -d_\lambda$; for $\kappa\in\mathrm{X}_*(T)$ with all conjugates $\leq\lambda$, since $\dim(S_\kappa\cap\mathrm{Gr}_G^{\lambda})\leq\langle\rho,\kappa+\lambda\rangle$, we see that $\Gamma_{\mathrm{c}}(j_\kappa^*i_{\lambda!}i_\lambda^*A)$ has tor-amplitude $\leq\langle 2\rho,\kappa+\lambda\rangle-d_\lambda=d_\kappa$, and so does $\Gamma_{\mathrm{c}}(j_\kappa^*A)$; by the exact same reasoning with S_κ^- along with Verdier duality, we also see that $\Gamma(j_\kappa^{-!}A)$ has tor-amplitude $\geq d_\kappa$; hyperbolic localization now shows that the above direct sum is actually a finite projective Λ-module concentrated on degree 0.

It remains to show that the above direct sum computes the perverse fiber functor. Since A has a $\mathrm{X}_*(T)$-filtration with graded pieces $j_{\kappa!}j_\kappa^*A$, so does $\Gamma(A)$ with $\Gamma_{\mathrm{c}}(j_\kappa^*A)$, but the parity of d_κ stays the same within one connected component of Gr_G, so the extensions are all split and the theorem follows. $\square$

4. Tannakian formalism

We record [FS24, Proposition VI.10.2] here. See there for a proof.

Proposition 4.1 (Tannakian reconstruction)**.** *Let $\mathcal{A}$ be a rigid symmetric monoidal category, and let $\mathcal{C}$ be a commutative $\mathcal{A}$-algebra category. Moreover, let*

$$F\colon \mathcal{C} \to \mathcal{A}$$

be a symmetric monoidal $\mathcal{A}$-linear conservative functor, such that $\mathcal{C}$ admits and F reflects coequalizers of F-split parallel pairs. Assume that $\mathcal{C}$ can be written as a filtered union of full subcategories $(\mathcal{C}_\mu)_{\mu\in\mathrm{M}}$, stable under coequalizers of F-split parallel pairs and the $\mathcal{A}$-action, such that $F|_{\mathcal{C}_i}$ is representable by some $X_i \in \mathcal{C}$.

Then

$$H := \operatorname*{colim}_{\mu\in\mathrm{M}} F(X_i)^\vee \in \mathsf{Ind}(\mathcal{A})$$

admits a natural structure as a bialgebra with commutative multiplication and associative comultiplication, and $\mathcal{C}$ is naturally equivalent to the symmetric monoidal category of H-comodules in $\mathcal{A}$. If $\mathcal{C}$ is rigid, then H is a Hopf algebra.

Therefore, once we can enhance the convolution and the fiber functor so that Proposition 4.1 applies to $\mathsf{Sat}(G;\Lambda) \to \mathsf{LocSys}(\mathrm{Spec}(k);\Lambda)$, we will get a local system of flat Hopf Λ-algebras or equivalently flat affine group Λ-schemes on $\mathrm{Spec}(k)$, which will be the *Langlands dual* of the reductive group G/k.

Remark 4.2 (semisimplicity)**.** Assume that Λ is a $\mathbb{Q}$-algebra. Once we know the Satake equivalence, representation theory will imply that $\mathsf{Sat}(G;\Lambda)$ is semisimple. Alternatively, one can see this fact directly from sheaf theory by a simple discussion on $\mathrm{Gr}^{\leq\mu}$ for quasi-minuscule μ followed by the decomposition theorem.

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