ARKHIPOV-BEZRUKAVNIKOV'S EQUIVALENCE III: CONCLUSION

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Abstract. We conclude the proof of Arkhipov and Bezrukavnikov's equivalence, i.e., that the functor $F\colon D^{\check{G}}(\widetilde{\mathcal{N}}) \to {}^f\mathcal{D}_I$ from Michael's talk is an equivalence. The essential surjectivity follows from the fact that the image of Wakimoto sheaves generate ${}^f\mathcal{D}_I$. The fully faithfulness is more involved, and requires two key technical inputs: 1) certain t-exactness properties of Gaitsgory's central sheaves, and 2) a certain quotient of $\mathcal{D}_I$ known as the regular quotient.

Let G be a reductive group over $\overline{\mathbb{F}}_p$ and let $\check{G}$ be the Langlands dual group over $\overline{\mathbb{Q}}_\ell$. Let $\mathcal{P}_I$ be the category of I-equivariant perverse sheaves on the affine flag variety $\mathcal{F}\ell := LG/I$. Let j_w be the embedding of the Schubert cell $\mathcal{F}\ell_w \hookrightarrow \mathcal{F}\ell$ and let $L_w := j_{w!*}\overline{\mathbb{Q}}_\ell[\ell(w)]$. Let ${}^f\mathcal{P}_I := \mathcal{P}/\langle L_w : w \notin {}^fW\rangle$ where ${}^fW \subset W$ is the set of minimal length representatives of left W_f-cosets in W. Our goal is to prove the following.

Theorem 0.1 (Arkhipov-Bezrukavnikov [AB09, Theorem 1])**.** *There is an equivalence of categories*

$$F\colon D^{\check{G}}(\widetilde{\mathcal{N}}) \simeq {}^f\mathcal{D}_I := D({}^f\mathcal{P}_I).$$

Although we will not prove it fully here, we will sketch some ideas and give detailed exposition on some technical ingredients. In particular we will see an application of the theory of highest weight categories, which is ubiquitous in representation theory.

1. Overview

1.1. **What has been done so far?** Gaitsgory's central functor $\mathcal{Z}$ and Wakimoto sheaves gives a functor $\operatorname{Rep}(\check{G} \times \check{T}) \to {}^f\mathcal{P}_I$. Michael enhanced this to a functor

$$F\colon D^{\check{G}}(\widetilde{\mathcal{N}}) \to {}^f\mathcal{D}_I.$$

Guido considered the pro-unipotent radical I_u^- of the negative Iwahori $I^- \subset L^+G$ and fixed a non-degenerate character $\psi\colon I_u^- \to \mathbb{G}_a$. He then defined the Iwahori-Whittaker category $\mathcal{P}_{IW}$ of (I_u^0, ψ)-equivariant perverse sheaves on $\mathcal{F}\ell$. For $w \in W$ we let $\mathcal{F}\ell^w$ be the corresponding I_u^--orbit in $\mathcal{F}\ell$ and consider the embedding $i_w\colon \mathcal{F}\ell^w \hookrightarrow \mathcal{F}\ell$. For each $w \in {}^fW$ there is a morphism $\psi_w\colon \mathcal{F}\ell_w \to \mathbb{G}_a$ given by $g \cdot wI \to \psi(g)$, and define the objects $\Delta_w := i_{w!}\psi_w^*\mathbb{AS}[\ell(w)]$ and $\nabla_w := i_{w*}\psi_w^*\mathbb{AS}[\ell(w)]$ of $\mathcal{P}_{IW}$. The category $\mathcal{D}_I := D(\mathcal{P}_I)$ acts on $\mathcal{D}_{IW} := D(\mathcal{P}_{IW})$ by convolution. Acting on the object ∇_0 produces a functor $Av_\psi\colon {}^f\mathcal{D} \to \mathcal{D}_{IW}$. Guido proved the following.

Theorem 1.2. *The functor $Av_\psi|_{\mathcal{P}_I}$ induces an equivalence ${}^f\mathcal{P}_I \simeq \mathcal{P}_{IW}$.*

Thus, it suffices to prove the following.

Theorem 1.3. *The functor $F_{IW} := Av_\psi \circ F$ induces an equivalence $D^{\check{G}}(\widetilde{\mathcal{N}}) \simeq \mathcal{D}_{IW}$.*

1.4. **Essential surjectivity.**

Lemma 1.5 ([AB09, Theorem 5])**.** *For any $\lambda \in \Lambda$, the strata $\mathcal{F}\ell_\lambda$ is open in the support of the Wakimoto sheaf J_λ, and $j_\lambda^* J_\lambda \simeq \underline{\overline{\mathbb{Q}}_\ell}[\ell(\lambda)]$.*

Proof. A computation in the Grothendieck group $K_0(\mathcal{D}_I) \simeq \mathbb{Z}[W]$ shows $[J_\mu] = [j_{\mu!}]$. Thus for any $w \in W$ the Euler characteristic of the stalk $\sum_i (-1)^i H^i(j_w^*(J_\lambda))$ is $(-1)^{\ell(\lambda)}\delta_{w,\lambda}$. □

Let κ denote the bijection $\Lambda \simeq {}^fW$ which takes a weight λ to the minimal length representative in the coset $W_f\lambda$.

Proposition 1.6 ([AB09, Lemma 22])**.** *The objects $\{Av_\psi(J_\lambda)\}_{\lambda\in\Lambda}$ generate the category $\mathcal{D}_{IW}$.*

Proof. By Lemma 1.5 the support of $Av_\psi(J_\lambda)$ is contained in the closure of $\mathcal{F}\ell^{\kappa(\lambda)}$. Moreover $j_\lambda^*(J_\lambda) \simeq j_\lambda^*(j_{\lambda!})$, since $Hom_{\mathcal{D}_I}(J_\lambda, j_{w*}) = 0$ for all $w \in W_f\lambda\backslash\{\lambda\}$ (see [Bez06, Lemma 11] for details). Thus $j^*_{\kappa(\lambda)}Av_\psi(J_\lambda)$ has rank 1. Now induction on the order on the strata shows $Av_\psi(J_\lambda)$ generate. ☐

Corollary 1.1. *The functor F_{IW} is essentially surjective.*

1.7. **Overview of the fully faithfulness.** Throughout, let us assume that $\check{G}$ has a faithful minuscule representation.[1] Equivalently, $\check{G}$ has no simple factors of type G_2, F_4, or E_8. This hypothesis will not be visible in our overview, but will be a key simplifying assumption for the following dimension formula.

For $\mathcal{F} \in \mathcal{D}_{IW}$ and $\mu \in \Lambda$, choose a point $x \in \mathcal{F}\ell^{\kappa(\mu)}$ and let $Stalk_\mu(\mathcal{F}) := i_x^*\mathcal{F}[-\dim \mathcal{F}\ell^{\kappa(\mu)}]$ and $coStalk_\mu(\mathcal{F}) := i_x^!\mathcal{F}[-\dim \mathcal{F}\ell^{\kappa(\mu)}]$.

Proposition 1.8 ([AB09, Proposition 7])**.** *For any representation V of $\check{G}$ and weight μ there are isomorphisms*

$$Stalk_\mu(Av_\psi(\mathcal{Z}_V)) \simeq coStalk_\mu(Av_\psi(\mathcal{Z}_V)) \simeq \overline{\mathbb{Q}}_\ell^{\oplus[\mu:V]} \tag{1.2}$$

where $[\mu : V]$ is the multiplicity of the weight μ in V.

Proof. By the Wakimoto filtration (discussed by Xingzhu) and since $[J_\mu] = [j_{\mu!}]$ as in Lemma 1.5,

$$[\mathcal{Z}_V] = \bigoplus_\mu [\mu : V] \cdot [J_\mu] = \bigoplus_\mu [\mu : V] \cdot [j_{\mu!}].$$

Applying F_{IW} gives $[Av_\psi(\mathcal{Z}_V)] = \bigoplus_\mu [\mu : V] \cdot [\Delta_\mu]$, and thus

$$\sum_i (-1)^i \dim H^i(Stalk_\mu(Av_\psi \mathcal{Z}_V)) = [\mu : V].$$

We will prove later (Proposition 2.1) that $Stalk_\mu(Av_\psi\mathcal{Z}_V)$ is concentrated in degree zero and the computation follows. ☐

Next, we introduce the *regular quotient* of the category $\mathcal{P}_{IW}$. To motivate this, let us pretend we knew $F_{IW}\colon D^{\check{G}}(\widetilde{\mathcal{N}}) \to \mathcal{D}_{IW}$ was an equivalence. Recall the Springer resolution $\widetilde{\mathcal{N}} \to \mathcal{N}$ is birational, and in particular is an isomorphism on the open $\check{G}$-orbit $\mathcal{N}_{reg} \subset \mathcal{N}$ of regular nilpotent elements in $\check{\mathfrak{g}}$. Therefore, there is a quotient functor $\mathrm{Coh}^{\check{G}}(\widetilde{\mathcal{N}}) \to \mathrm{Coh}^{\check{G}}(\mathcal{N}_{reg}) = \mathrm{Rep}(Z_{\check{G}}(e))$ where $Z_{\check{G}}(e) \subset \check{G}$ is the centralizer of a regular nilpotent element $e \in \mathcal{N}_{reg}$. A natural question is: what does this quotient correspond to on the constructible side?

Note that $Z_{\check{G}}(e)$ is an extension of $Z_{\check{G}}$ by a unipotent group, so the irreducible objects are parametrized by characters of $Z_{\check{G}}$. Such characters are parametrized by the algebraic fundamental group $\pi_1^{alg}(G)$, which bijects with the elements of W of length zero. Thus, a natural guess of the constructible side is the Serre quotient $\mathcal{P}_I^{reg} := \mathcal{P}_I/\langle L_w : \ell(w) > 0\rangle$.

Although we can't immediately prove $\mathcal{D}_I^{reg} := D(\mathcal{P}_I^{reg})$ is equivalent to $D(\mathrm{Rep}(Z_{\check{G}}(e)))$, we can prove an approximation to this. Let $\widetilde{\mathcal{P}}_I^{reg}$ be the full subcategory of $\mathcal{P}_I^{reg}$ consisting of all subquotients of the composition

$$\mathrm{Rep}(\check{G}) \xrightarrow{\mathcal{Z}} \mathcal{P}_I \to \mathcal{P}_I^{reg}.$$

The category $\mathcal{P}_I^{reg}$ will inherit a monoidal structure from $\mathcal{P}_I$ (Lemma 3.1), and we can prove the following Tannakian reconstruction result.

[1] A representation is *minuscule* if the Weyl group W_f acts transitively on the weights.

Proposition 1.9 ([Bez04, §5])**.** *There exists a subgroup $H \subset Z_{\check{G}}(e)$ and an equivalence of monoidal categories $\mathrm{Rep}(H) \simeq \widetilde{\mathcal{P}}_I^{reg}$, fitting into a commutative diagram*

$$\begin{array}{ccc} \mathrm{Rep}(\check{G}) & \xrightarrow{\mathcal{Z}} & \mathcal{P}_I \\ {\scriptstyle res_H^{\check{G}}}\downarrow & & \downarrow \\ \mathrm{Rep}(H) & \xrightarrow{\sim} & \widetilde{\mathcal{P}}_I^{reg}. \end{array} \tag{1.3}$$

In fact, we will see later that the nilpotent element e becomes the monodromy operator on $\mathcal{Z}$ on the constructible side.

Remark. In fact $\widetilde{\mathcal{P}}_I^{reg} = \mathcal{P}_I^{reg}$ and $H = Z_{\check{G}}(e)$, as discussed in [ALWY25, Proposition 10.8].

Now, we finally move to the proof that F_{IW} is an equivalence. We already saw essential surjectivity (Corollary 1.1), so it suffices to check F_{IW} is fully faithful. Recall from Michael's talk that there are two generating subsets of $D^{\check{G}}(\widetilde{\mathcal{N}})$, namely $\{\mathcal{O}(\lambda)\}_{\lambda\in\Lambda}$ and $\{\mathcal{O}(-\lambda)\otimes V_\mu\}_{\lambda,\mu\in\Lambda^+}$. Thus the key computation is to check that for any $V \in \mathrm{Rep}(\check{G})$ and any $\mu \in \Lambda$ the map

$$Hom_{D^{\check{G}}(\widetilde{\mathcal{N}})}(V\otimes\mathcal{O}, \mathcal{O}(\mu)) \to Hom_{\mathcal{D}_{IW}}(Av_\psi(\mathcal{Z}_V), Av_\psi(J_\mu)) \tag{1.4}$$

is an isomorphism. By twisting by line bundles we may assume μ is dominant. Results of [KLT99] show that the left-hand side is concentrated in degree zero and

$$Hom_{Coh^{\check{G}}(\widetilde{\mathcal{N}})}(V\otimes\mathcal{O}, \mathcal{O}(\mu))$$

has dimension $[\mu : V]$. On the other hand by (1.2) the right-hand side equals

$$Hom_{\mathcal{D}_{IW}}(Av_\psi(\mathcal{Z}_V), Av_\psi(J_\mu)) \simeq Hom_{\mathcal{D}_{IW}}(Av_\psi(\mathcal{Z}_V), Av_\psi(j_{\mu*})) \simeq Hom(Stalk_\mu(Av_\psi(\mathcal{Z}_V)), \overline{\mathbb{Q}}_\ell)$$

which is of dimension $[\mu : V]$ and concentrated in degree zero. Furthermore, by passing to the regular quotient we see (1.4) is injective. The equivalence follows.

2. Central sheaves are tilting

The main goal in this section is to prove the following technical input to the dimension formula (1.2).

Proposition 2.1 ([AB09, Theorem 7])**.** *Let $V \in \mathrm{Rep}(\check{G})$, and let $\mathcal{Z}_V \in \mathcal{P}_I$ be the corresponding central sheaf. Then for any $w \in {}^fW$, the complexes $i_w^! Av_\psi(\mathcal{Z}_V)$ and $i_w^* Av_\psi(\mathcal{Z}_V)$ are concentrated in homological degree $-\dim(\mathcal{F}\ell^w)$.*

The proof of Proposition 2.1 proceeds by proving that the statement for minuscule representations (Lemma 2.4), then showing that if the statement holds for V_1 and V_2 then it also does for $V_1 \otimes V_2$ (Lemma 2.6). Under our assumption any irreducible representation of V is found in a tensor power of a minuscule representation, from which Proposition 2.1 follows.[2]

First, we use the following general fact about perverse sheaves from [BBM04, Proposition 1.3].

Proposition 2.2. *Let $\mathcal{F}$ be a perverse sheaf on $X = \bigsqcup_\nu X_\nu$, and assume the embedding $i_\nu : X_\nu \hookrightarrow X$ is affine. The following are equivalent:*

(1) for each ν the complexes $i_\nu^*\mathcal{F}$ and $i_\nu^!\mathcal{F}$ are perverse, i.e., in homological degree $-\dim(X_\nu)$; and
(2) $\mathcal{F}$ is a successive extension of $i_{\nu*}M_\nu$ with M_ν perverse, and $i_{\mu!}N_\mu$ with N_μ perverse.

[2] Arkhipov and Bezrukavnikov also prove the analogous statement for *quasi-minuscule* representations, which exist for any group. However this involves a lemma of Gaitsgory and is technically more complicated so we omit it here.

Proof. That (2) implies (1) follows from

$$i_\nu^! i_{\mu *} M = i_\nu^* i_{\mu !} M = \begin{cases} M & \text{if } \mu = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Next, assume $\mathcal{F}$ satisfies (1), and choose a closed filtration $X \supset X_1 \supset \cdots \supset X_n = \emptyset$ such that each $X_i \backslash X_{i+1}$ is a single stratum. We will just prove that $\mathcal{F}$ is a successive extension of $i_{\nu *} M_\nu$ with M_ν perverse, since the analogous statement for shriek extensions is argued similarly. We will inductively prove that for the open immersion $j_k \colon X \backslash X_k \hookrightarrow X$, the pullback $j_k^* \mathcal{F}$ is a successive extension of $j_k^* i_{\nu *} M_\nu$ where M_ν is perverse (so $k = n$ is the statement we want). We assume the statement for k, and argue that $j_{k+1}^* \mathcal{F}$ is such a successive extension. By assumption $X_k \backslash X_{k+1}$ is a single stratum X_ν. Then $j_{k+1}^* \mathcal{F}$ sits in an exact triangle

$$i_{\nu *} i_\nu^! \mathcal{F} \to j_{k+1}^* \mathcal{F} \to j_{k *} j_k^* \mathcal{F}.$$

Here $i_\nu^! \mathcal{F}$ is perverse by (1), and by our inductive assumption $j_{k*} j_k^* \mathcal{F}$ has a filtration by $i_{\nu *} M_\nu$ where M_ν is perverse. □

Definition 2.3. When one of the equivalent conditions of Proposition 2.2 hold, $\mathcal{F}$ is said to be *tilting*.

We prove Proposition 2.1 in the following two steps.

Lemma 2.4 ([AB09, Lemma 26])**.** *If V is a minuscule representation of $\check{G}$, then $Av_\psi(\mathcal{Z}_V)$ is tilting.*

Remark. There is a more general statement for quasi-minuscule representations, which is necessary to prove Theorem 1.3 for arbitrary $\check{G}$. However it is significantly more technical so we omit it here.

To prove Lemma 2.4, we need the following sub-lemma.

Lemma 2.5 ([AB09, Lemma 28])**.** *For any $w_f \in W_f$, $\lambda \in \Lambda$, and $V \in \mathrm{Rep}(\check{G})$, there is an isomorphism $Stalk_\lambda(\mathcal{Z}_V) \simeq Stalk_{w_f(\lambda)}(\mathcal{Z}_V)$.*

Proof. It suffices to check the statement when $w = s \in W_f$ is a simple reflection. Then $Av_\psi(L_s) = 0$. Indeed, we know L_s is the constant sheaf on the strata $I_s/I \subset \mathcal{F}\ell$ where $I_s \supset I$ is the parahoric subgroup corresponding to the simple reflection s. But then $Av_\psi(L_s)$ is the pullback of the push-forward of Δ_0 along $\mathcal{F}\ell \to LG/I_s$. The pushforward is zero since ψ is no longer trivial on the I_u^--stabilizer $I_u^- \cap I_s$ of the orbit I_s. By central properties of $\mathcal{Z}_V$,

$$Av_\psi(\mathcal{Z}_V) \star L_s = (\Delta_0 \star L_s) \star \mathcal{Z}_V = 0.$$

Since there is an exact triangle $L_s \to j_{s*} \to j_{1*}$, we see that

$$Av_\psi(\mathcal{Z}_V) \star j_{s*} \simeq Av_\psi(\mathcal{Z}_V).$$

Here, note that $Av_\psi(\mathcal{Z}_V) \star j_{s*}$ is computed via a pushforward along $LG \times^I IsI/I \to \mathcal{F}\ell$. When $\kappa(\lambda)s > \kappa(\lambda)$ this restricts to an isomorphism on $\mathcal{F}\ell^{\kappa(\lambda)} \subset \mathcal{F}\ell$ and the claim follows. □

Proof of Lemma 2.4. We only check that $Stalk_\mu(\mathcal{F})$ is concentrated in homological degree zero for all $\mu \in \Lambda$, since the argument for costalks is the same.

Let λ be a minuscule weight and let $V = V_\lambda$. The support of $\mathcal{Z}_V$ is contained in the pre-image of Schubert cell $\overline{\mathcal{G}r}_\lambda = \mathcal{G}r_\lambda$ under the projection $\mathcal{F}\ell \to \mathcal{G}r$. In particular, the only strata that lie in $W_f \lambda W_f \subset W$ appear. Here $\mathcal{F}\ell_\lambda$ is open in the support of $\mathcal{Z}_V$ and $j_\lambda^* \mathcal{Z}_V \simeq \underline{\overline{\mathbb{Q}}_\ell}[\ell(\lambda)]$, which implies

$$Stalk_{w_0 \lambda}(Av_\psi(\mathcal{Z}_V)) \simeq \overline{\mathbb{Q}}_\ell.$$

By Lemma 2.5 it also follows that, for any $w_f \in W_f$,

$$Stalk_{w_f \lambda}(Av_\psi(\mathcal{Z}_V)) \simeq \overline{\mathbb{Q}}_\ell.$$

Since the support of $Av_\psi(\mathcal{Z}_V)$ consists of strata $\mathcal{F}\ell^{\kappa(\mu)}$ where $\mu \in W_f(\lambda)$, the claim follows. □

Lemma 2.6 ([AB09, Lemma 25]). *If V_1 and V_2 are representations of $\check{G}$ such that $Av_\psi(\mathcal{Z}_{V_i})$ are tilting, then so is $Av_\psi(\mathcal{Z}_{V_1\otimes V_2})$.*

To prove Lemma 2.6 we need the following sub-lemma.

Lemma 2.7 ([AB09, Sublemma 2]). *For any $v \in {}^fW$ and $w \in W$ and $\lambda \in \Lambda$,*

$$Stalk_\lambda(\Delta_v \star j_{w!}) \in D(\mathrm{Vect})^{\geq 0}.$$

Proof. The statement is equivalent to $\Delta_v \star j_{w!} \in \langle \Delta_u[i] : i \leq 0, u \in {}^fW\rangle$. Furthermore since $\Delta_w \simeq \Delta_0 \star j_{x!}$ for any $w \in {}^fW \cap W_f x$, we further reduce to the statement

$$j_{v!} \star j_{w!} \in \langle j_{u!}[i] : i \leq 0, u \in W\rangle. \tag{2.1}$$

It suffices to check (2.1) when $w = s$ is a simple reflection. If $vs > v$ then $j_{v!} \star j_{s!} \simeq j_{vs!}$ and if $vs < v$ then there is an exact triangle[3]

$$j_{v!} \oplus j_{v!}[-1] \to j_{v!} \star j_{s!} \to j_{vs!}.$$

In either case (2.1) holds. ☐

Proof of Lemma 2.6. By assumption we know $Av_\psi(\mathcal{Z}_{V_1})$ is a successive extension of standard objects Δ_w, with $w \in {}^fW$. Thus

$$Av_\psi(\mathcal{Z}_{V_1\otimes V_2}) = Av_\psi(\mathcal{Z}_{V_1}) \star \mathcal{Z}_{V_2}$$

is a successive extension of $\mathcal{F} := \Delta_w \star \mathcal{Z}_{V_2} = \Delta_0 \star \mathcal{Z}_{V_2} \star j_{w!}$. By the convolution exactness of central sheaves we know $\mathcal{F}$ is perverse. Thus by the definition of perverse sheaves $Stalk_\lambda(\mathcal{F}) \in D(\mathrm{Vect})^{\leq 0}$ and by Lemma 2.7 we see $Stalk_\lambda(\mathcal{F}) \in D(\mathrm{Vect})^{\geq 0}$. Thus the stalks of $Av_\psi(\mathcal{Z}_{V_1\otimes V_2})$ are concentrated in degree zero. The same argument for co-stalks shows $Av_\psi(\mathcal{Z}_{V_1\otimes V_2})$ is tilting. ☐

Finally, we may conclude our proof that central sheaves are tilting.

Proof of Proposition 2.1. Let V be a direct sum of the minuscule representations of $\check{G}$. Then V is a faithful representation, adnd hence $\check{G}$ is a closed subgroup of $\mathrm{End}(V)$. By Peter-Weyl any irreducible representation of $\check{G}$ occurs in the ring of functions $\mathcal{O}(\check{G})$, which is a quotient of the ring of functions on $\mathrm{End}(V)$. Moreover since V is self-dual we know $\mathrm{End}(V) \simeq V^{\otimes 2}$. Thus any irreducible representation of $\check{G}$ is a summand of some $\mathrm{Sym}^n(V^{\otimes 2})$, and in particular of some tensor power $V^{\otimes 2n}$. Thus the claim follows from Lemma 2.4 and Lemma 2.6. ☐

3. Regular quotient

Our goal is to show Proposition 1.9. Let us first construct a monoidal structure on the quotient $\mathcal{P}_I^{reg} := \mathcal{P}_I / \langle L_w : \ell(w) > 0\rangle$.

Lemma 3.1 ([ALWY25, Proposition 8.1]). *The monoidal structure ${}^pH^0(- \star -)$ descends to an exact monoidal structure on the quotient $\mathcal{P}_I^{reg}$.*

Proof. It suffices to check that $\langle L_w : \ell(w) > 0\rangle$ is an ideal under the monoidal structure, i.e., that for any $w \in W$ such that $\ell(w) > 0$ and simple reflection s,

$${}^pH^0(L_w \star L_s) \in \langle L_v : \ell(v) > 0\rangle.$$

But if $I_s \supset I$ is the parahoric corresponding to s then L_s is a constant sheaf on $I_s/I \subset \mathcal{F}\ell$, hence $L_w \star L_s$ is right I_s-equivariant. Thus so is ${}^pH^0(L_w \star L_s)$, which means it must lie in $\langle L_v : vs < v\rangle$.

To check exactness, we must show that the higher perverse cohomologies of $L_x \star L_y$ lies in $\langle L_v : \ell(v) > 0\rangle$ for any $x, y \in W$. If either x or y has nonzero length our previous argument applies, so we may assume $\ell(x) = \ell(y) = 0$. But then $L_x \star L_y = L_{xy}$ is already perverse. ☐

[3] This is a categorification of the identity $T_vT_s = (q-1)T_v + qT_{vs}$ in the affine Hecke algebra. It can br proved by convolving the exact triangle $j_{s!} \oplus j_{s!}[-1] \to j_{s!} \star j_{s!} \to j_{1!}$ (which is a computation on $\mathbb{P}^1$) by $j_{vs!}$.

Consider the central monoidal functor

$$\mathcal{Z}^{reg}\colon \operatorname{Rep}(\check{G}) \xrightarrow{\mathcal{Z}} \mathcal{P}_I \to \mathcal{P}_I^{reg}.$$

Let $\widetilde{\mathcal{P}}_I^{reg}$ be the full subcategory of $\mathcal{P}_I^{reg}$ consisting of all subquotients of the composition. Recall that Gaitsgory's functor $\mathcal{Z}$ was defined in terms of nearby cycles, hence carries a monodromy operator $\mathfrak{n}_V\colon \mathcal{Z}_V \to \mathcal{Z}_V$. We consider the induced nilpotent endomorphism $\mathfrak{n}_V^{reg}\colon \mathcal{Z}_V^{reg} \to \mathcal{Z}_V^{reg}$.

Proposition 3.2 ([Bez04], [ALWY25, Proposition 8.3])**.** *There exists a nilpotent element $e \in \check{\mathfrak{g}}$, a subgroup $H \subset Z_{\check{G}}(e)$, and an equivalence of monoidal categories $\operatorname{Rep}(H) \simeq \widetilde{\mathcal{P}}_I^{reg}$, fitting into a commutative diagram* (1.3)*. For $V \in \operatorname{Rep}(\check{G})$, the equivalence intertwines the endomorphism e on $res_H^{\check{G}} V$ with the monodromy $\mathfrak{n}^{reg}$ on $\mathcal{Z}_V^{reg}$.*

Proof. From general Tannakian formalism, there exists a group $H \subset \check{G}$ and an equivalence $\operatorname{Rep}(H) \simeq \widetilde{\mathcal{P}}_I^{reg}$ making (1.3) commute. Recall from Bhargav's talk that the monodromy operator is monoidal, in the sense that for any two representations V and W of $\check{G}$,

$$\mathfrak{n}_{V\otimes W}^{reg} = \mathfrak{n}_V^{reg} \otimes 1 + 1 \otimes \mathfrak{n}_W^{reg}.$$

By the Tannakian formalism this corresponds to a tensor endomorphism of $For_H^{\check{G}}$, which corresponds to an element $e \in \check{\mathfrak{g}}$ such that $H \subset Z_{\check{G}}(e)$. □

Now, the only thing missing from Proposition 1.9 is to check the nilpotent element e in Proposition 3.2 is regular. This argument involves the weight filtration on mixed sheaves [PNA, §6.5.8], [ALWY25, Proposition 8.6], which requires a little bit of preparation.

Lemma 3.3 ([Del80, Proposition 1.6.1])**.** *Let N be a nilpotent endomorphism of an object V of an abelian category. Then there exists a unique finite*[4] *filtration*

$$\cdots \subset \operatorname{Fil}^{-1} V \subset \operatorname{Fil}^0 V \subset \operatorname{Fil}^1 V \subset \cdots$$

such that $N(\operatorname{Fil}^i V) \subset \operatorname{Fil}^{i-2} V$ and N^k induces an isomorphism $\operatorname{gr}^k V \simeq \operatorname{gr}^{-k} V$ for all $k \geq 0$.

Proof. Let d be an integer such that $N^{d+1} = 0$. First note that for $k > d$ the isomorphism $\operatorname{gr}^k V \simeq \operatorname{gr}^{-k} V$ is given by $N^k = 0$, hence

$$\operatorname{Fil}^d V = \operatorname{Fil}^{d+1} V = \cdots \tag{3.1}$$

which must equal V by finiteness, and

$$\operatorname{Fil}^{-d-1} V = \operatorname{Fil}^{-d-2} V = \cdots \tag{3.2}$$

which must equal 0 by finiteness.

Now let us induct on d. If $d = 0$ (i.e., $N = 0$) by (3.1) and (3.2) the only filtration that could work is the trivial one

$$\operatorname{Fil}^i V = \begin{cases} V & \text{if } i \geq 0 \\ 0 & \text{if } i < 0. \end{cases}$$

This clearly satisfies the conditions.

Next, set

$$\operatorname{Fil}^{-d-1} V = 0 \subset \operatorname{Fil}^{-d} V = \operatorname{im}(N^d), \quad \operatorname{Fil}^{d-1} V = \ker(N^d) \subset \operatorname{Fil}^d V = V,$$

so that N^d induces the standard isomorphism $V/\ker(N^d) \simeq \operatorname{im}(N^d)$. This is furthermore our unique choice, since if there is an isomorphism

$$N^d\colon V/\operatorname{Fil}^{d-1} V \simeq \operatorname{Fil}^{-d} V$$

then for injectivity we need $\operatorname{Fil}^{d-1} V = \ker(N^d)$ and for surjecivity we need $\operatorname{Fil}^{-d} V = \operatorname{im}(N^d)$. But then $N^d = 0$ on the quotient $\operatorname{Fil}^{d-1} V/\operatorname{Fil}^{-d} V$, and we may apply our inductive hypothesis. □

[4] i.e., there exists a $N \gg 0$ such that $\operatorname{Fil}^{-N} V = 0$ and $\operatorname{Fil}^N V = V$

A filtration as in Lemma 3.3 is called the *Jacobson–Morozov–Deligne filtration*.

Example 3.4. By Jordan normal form, any nilpotent endomorphism of a finite-dimensional vector space is a direct sum of endomorphisms of the form e acting on $V = k[e]/e^{d+1}$. Then the Jacobson-Morozov-Deligne filtration is

$$\begin{aligned}\operatorname{Fil}^{-d-1} V = 0 &\subset \operatorname{Fil}^{-d} V = \operatorname{Fil}^{1-d} V = e^d V \\ &\subset \operatorname{Fil}^{2-d} V = \operatorname{Fil}^{3-d} V = e^{d-1} V \\ &\subset \cdots \\ &\subset \operatorname{Fil}^{d-2} V = \operatorname{Fil}^{d-1} V = eV \subset \operatorname{Fil}^{d} V = V.\end{aligned}$$

In particular for this vector space, $\dim(\operatorname{gr}^0 V) + \dim(\operatorname{gr}^1 V) = 0$. Generally, the number of Jordan blocks of a nilpotent endomorphism e acting on V is

$$\dim(\ker e) = \dim(\operatorname{gr}^0 V) + \dim(\operatorname{gr}^1 V). \tag{3.3}$$

Proof of Proposition 1.9. Let V be a representation of $\check{G}$. Then the Satake sheaf $\mathcal{S}_V \in \operatorname{Perv}_{L^+G}(\mathcal{G}r_G)$ is a direct sum of IC sheaves, hence it is a semisimple mixed perverse sheaf. Now by applying the central functor gives a mixed perverse sheaf $\mathcal{Z}_V^{mix}$ on $I\backslash\mathcal{F}\ell$.

A result of Gabber [BiB93, §5.1] says the Jacobson-Morosov-Deligne filtration of the nilpotent endomorphism $\mathfrak{n}_V$ of $\mathcal{Z}_V^{mix}$ matches the weight filtration on $\mathcal{Z}_V^{mix}$.

Moreover by the uniqueness the Jacobson-Morosov-Deligne filtration is functorial, hence the image of the weight filtration under the functor $\mathcal{P}_I \to \widetilde{\mathcal{P}}_I^{reg} \simeq \operatorname{Rep}(H)$ is the Jacobson-Morosov filtration of e acting on V.

Note that the functor $\mathcal{P}_I \to \operatorname{Rep}(H)$ kills the objects L_w when $\ell(w) > 0$, and when $\ell(w) = 0$ since $L_w \in \mathcal{P}_I$ is invertible, its image has dimension one. Thus, the i-th graded piece of the Jacobson-Morozov-Deligne filtration of e acting on V is

$$\sum_{\ell(w)=0} [\operatorname{gr}_i^W(\mathcal{Z}_V^{mix}) : L_w^{mix}].$$

The point is that the sum may now be computed combinatorially. By [PNA, Lemma 5.3.2] the Grothendieck group of $\mathcal{P}_I^{mix}$ is the affine Hecke algebra $\mathcal{H}_v(W)$ and the homomorphism $\eta\colon \mathcal{H}_v(W) \to \mathbb{Z}[v^{\pm 1}]$ sending the standard basis H_w to $(-v)^{\ell(w)}$ is given by sending the class of a mixed perverse sheaf $[\mathcal{F}]$ to

$$\sum_{i\in\mathbb{Z}} \dim(\operatorname{gr}_i^W \mathcal{F}) v^i.$$

The homomorphism acts on the canonical basis (which correspond to $[L_w^{mix}]$) by

$$\eta(\underline{H}_w) = \begin{cases} 1 & \text{if } \ell(w) = 0 \\ 0 & \text{if } \ell(w) > 0, \end{cases}$$

and sends the Bernstein generators θ_λ (which correspond to Wakimoto sheaves) to $v^{\langle\lambda,2\rho\rangle}$. Thus the isomorphism $K_0(\mathcal{P}_I^{mix}) \simeq \mathcal{H}_v(W)$ sends the central sheaf $\mathcal{Z}_V^{mix}$ to $\sum_\lambda [V:\lambda]\theta_\lambda$. On the other hand,

$$[\mathcal{Z}_V^{mix}] = \sum_{\ell(w)=0} [\mathcal{Z}_V^{mix} : L_w^{mix}] \cdot [L_w^{mix}].$$

By looking at the image under η, we conclude

$$\sum_{i\in\mathbb{Z}} \sum_{\ell(w)=0} [\operatorname{gr}_i^W \mathcal{Z}_V^{mix} : L_w^{mix}] \cdot v^i = \sum_\lambda [V:\lambda] v^{\langle\lambda,2\rho\rangle}.$$

By combining the considerations with (3.3), we conclude that

$$\dim(V^e) = \sum_{\langle\lambda,2\rho\rangle\in\{0,1\}} \dim[V:\lambda]. \tag{3.4}$$

In particular when $V = \check{\mathfrak{g}}$ is the adjoint representation we obtain the dimension of $\check{\mathfrak{g}}^e$ is the rank of $\check{T}$, i.e., e is regular. ☐

Remark. That equation (3.4) holds for regular nilpotent e is a purely representation theoretic fact. I will give an elementary argument here, just for fun. It is known that e can be completed into a homomorphism $\mathfrak{sl}_2 \to \check{\mathfrak{g}}$ such that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is sent to e and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is sent to $2\rho \in \check{\mathfrak{g}}$. Then (3.4) follows from the observation that for any representation V of $\mathfrak{sl}_2$, the dimension of V^e is $\dim(V^{h=0}) + \dim(V^{h=1})$, analogous to (3.3).

References

[AB09] Sergey Arkhipov and Roman Bezrukavnikov, *Perverse sheaves on affine flags and Langlands dual group*, Israel J. Math. **170** (2009), 135–183, With an appendix by Bezrukavnikov and Ivan Mirković. MR 2506322

[ALWY25] Johannes Anschütz, João Lourenço, Zhiyou Wu, and Jize Yu, *Gaitsgory's central functor and the Arkhipov-Bezrukavnikov equivalence in mixed characteristic*, 2025.

[BBM04] A. Beilinson, R. Bezrukavnikov, and I. Mirković, *Tilting exercises*, Mosc. Math. J. **4** (2004), no. 3, 547–557, 782. MR 2119139

[Bez04] Roman Bezrukavnikov, *On tensor categories attached to cells in affine Weyl groups*, Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math., vol. 40, Math. Soc. Japan, Tokyo, 2004, pp. 69–90. MR 2074589

[Bez06] ———, *Cohomology of tilting modules over quantum groups and t-structures on derived categories of coherent sheaves*, Invent. Math. **166** (2006), no. 2, 327–357. MR 2249802

[BiB93] A. Be˘ ilinson and J. Bernstein, *A proof of Jantzen conjectures*, I. M. Gelfand Seminar, Adv. Soviet Math., vol. 16, Part 1, Amer. Math. Soc., Providence, RI, 1993, pp. 1–50. MR 1237825

[Del80] Pierre Deligne, *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 52, 137–252. MR 601520

[KLT99] Shrawan Kumar, Niels Lauritzen, and Jesper Funch Thomsen, *Frobenius splitting of cotangent bundles of flag varieties*, Invent. Math. **136** (1999), no. 3, 603–621. MR 1695207

[PNA] Simon Riche Pramod N. Achar, *Central sheaves on the affine flag variety.*

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