INTRODUCTION TO BEZRUKAVNIKOV'S EQUIVALENCE

KENTA SUZUKI

Abstract. The goal of the seminar will be to prove the geometric Satake equivalence and Bezrukavnikov's equivalence, which are instances of the local geometric Langlands conjecture. We motivate these equivalences from the local Langlands conjecture, which attempts to understand the representation theory of p-adic groups $G(\mathbb{Q}_p)$ in terms of Galois representations into the Langlands dual group $\check{G}$.

Bezrukavnikov's equivalence is a theorem in *local geometric Langlands*. What do each of these words mean?

- *local*: we are dealing with local fields such as $\mathbb{Q}_p$ or $\mathbb{F}_p(\!(t)\!)$. Geometrically, this means working with the formal disk $D = \operatorname{Spec} \mathbb{C}[\![t]\!]$ and its puncture $D^\times = \operatorname{Spec} \mathbb{C}(\!(t)\!)$. This is in contrast to *global* Langlands, where we deal with number fields such as $\mathbb{Q}$, function fields such as $\mathbb{F}_p(t)$, or smooth projective curves $X/\mathbb{C}$.
- *geometric*: this is a code word for categorical. There are many isomorphisms or bijections known classically, and our goal will be to find a category equivalence which underlies them.
- *Langlands*: this is a general framework in which the representation theory of G can be understood in terms of the Langlands dual group $\check{G}$.

1. Local Langlands Conjecture

Warning: *Since this section is for motivation, everything is only approximately true!*

Let F be a local field with residue characteristic p (i.e., a finite extension of $\mathbb{F}_p(\!(t)\!)$ or $\mathbb{Q}_p$) and ring of integers $\mathcal{O}$, with residue field k of size q. Let $G/\mathcal{O}$ be a split reductive group. The Local Langlands Correspondence attempts to understand smooth representations of $G(F)$ in terms of the Langlands dual group $\check{G}$.

Conjecture 1.1 (Local Langlands Conjecture)**.** There is a certain group WD_F which is a small modification of the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ and there is (roughly) a bijection

$$\{\text{irreducible smooth representations of } G(F)\} \simeq \{\text{homomorphisms } \mathrm{WD}_F \to \check{G}(\mathbb{C})\}. \tag{1.2}$$

Here, if a representation V of $G(F)$ is *smooth* there exists some compact open subgroup $K \subset G(F)$ such that $V^K \neq 0$. For a fixed compact open subgroup $K \subset G(F)$, the irreducible representations of $G(F)$ with a K-fixed vector are in bijection with irreducible modules over the *Hecke algebra* $\mathbb{C}[K\backslash G(F)/K]$. Now there should (roughly) be a quotient Γ of WD_F corresponding to the subgroup K and the bijection (1.2) should restrict to a bijection

$$\{\text{irreducible modules of } \mathbb{C}[K\backslash G(F)/K]\} \simeq \{\text{homomorphisms } \Gamma \to \check{G}(\mathbb{C})\}. \tag{1.3}$$

We expect Conjecture 1.1 to become more and more difficult as K becomes smaller. Although the conjecture is open generally, we will discuss two known instances of the bijection (1.3): when $K = G(\mathcal{O})$ and $K = I$, the *Iwahori subgroup*.

2. Unramified Local Langlands

The easiest instance of local Langlands (i.e., when K is as large as possible) is the *unramified* local Langlands, when $K = G(\mathcal{O})$. In this case, the spherical Hecke algebra $\mathcal{H}_{\mathrm{sph}} := \mathbb{C}[G(\mathcal{O})\backslash G(F)/G(\mathcal{O})]$ admits the following description.

Proposition 2.1 (Satake)**.** *There is an isomorphism*

$$\mathcal{H}_{\mathrm{sph}} \simeq R(\check{G}), \tag{2.1}$$

where $R(\check{G})$ is the representation ring of $\check{G}$, i.e., $K_0(\mathrm{Rep}(\check{G}))$.

As a consequence, we deduce the unramified local Langlands.

Corollary 2.2. *There is a bijection between:*

- *irreducible representations of $G(F)$ with a $G(\mathcal{O})$-fixed vector; and*
- *semisimple elements $s \in \check{G}$ up to conjugation (the* Satake *parameter).*

Under the philosophy of geometric Langlands, we hope to find a categorical equivalence which recovers (2.1) upon passing to K-groups. A natural categorification of the left-hand side (at least in the equal characteristic case) is the category of perverse sheaves

$$\mathrm{Perv}(L^+G\backslash LG/L^+G),$$

where $LG(R) := G(R(\!(t)\!))$ is the loop group and $L^+G(R) := G(R[\![t]\!])$ is the positive loop group. On the other hand, the right-hand side naturally categorifies to the category of representations $\mathrm{Rep}(\check{G})$. Thus we would hope for an equivalence between these two categories. In fact, since (2.1) is an isomorphism of *commutative rings*, not just vector spaces, we hope that both categories carry a symmetric monoidal structure and the equivalence is symmetric monoidal. The Geometric Satake equivalence realizes our hope.

Theorem 2.2 (Mirković-Vilonen [MV07], Ginzburg [Gin00], Lusztig)**.** *There is a symmetric monoidal equivalence*

$$\mathrm{Perv}(L^+G\backslash LG/L^+G) \simeq \mathrm{Rep}(\check{G}), \tag{2.3}$$

where the left-hand side carries the convolution monoidal structure and the right-hand side carries the usual monoidal structure.

Remark 2.3. The theorem provides a "canonical" definition of the Langlands dual $\check{G}$, rather than via the combinatorics of root datum.

Remark 2.4. For the purposes of categorifying local Langlands for $\mathbb{F}_p(\!(t)\!)$ we should let G be over characteristic p. However, the equivalence (2.3) is not very sensitive to the characteristic. In the seminar we will try to work in characteristic zero as long as possible for simplicity.

For those of us derived-minded people, we may wonder what the entire *derived* category of constructible sheaves $D_{\mathrm{cons}}(L^+G\backslash LG/L^+G)$ looks like.

Theorem 2.5 (Bezrukavnikov-Finkelberg [BF08])**.** *There is a symmetric monoidal equivalence*

$$D_{\mathrm{cons}}(L^+G\backslash LG/L^+G) \simeq D^b_{\mathrm{coh}}\big((0 \times^{\mathbb{L}}_{\check{\mathfrak{g}}} 0)/\check{G}\big), \tag{2.4}$$

where the left-hand side carries a monoidal structure by convolution.

Remark 2.6. How could we have guessed $(0 \times^{\mathbb{L}}_{\check{\mathfrak{g}}} 0)/\check{G}$ would appear on the left-hand side? In geometric Langlands, representations of $G(\mathbb{Q}_p)$ are replaced by sheaves on the stack of G-bundles and Galois representations into $\check{G}$ are replaced by $\check{G}$-local systems. Consider the ravioli space $\mathrm{Rav} = D \cup_{D^\times} D$ where $D = \mathrm{Spec}\,\mathbb{C}[\![t]\!]$ is the formal disk and $D^\times = \mathrm{Spec}\,\mathbb{C}(\!(t)\!)$ is its puncture. Then, the stack $\mathrm{Bun}_G(\mathrm{Rav})$ of G-bundles on R is isomorphic to $L^+G\backslash LG/L^+G$ and the stack $\mathrm{LS}_{\check{G}}(\mathrm{Rav})$ of $\check{G}$-local systems on R is isomorphic to $\mathrm{pt}/\check{G} \times_{\check{G}/\check{G}} \mathrm{pt}/\check{G} \simeq (0 \times^{\mathbb{L}}_{\check{\mathfrak{g}}} 0)/\check{G}$. Thus (2.4) is equivalent to

$$D_{\mathrm{cons}}\big(\mathrm{Bun}_G(\mathrm{Rav})\big) \simeq D^b_{\mathrm{coh}}\big(\mathrm{LS}_{\check{G}}(\mathrm{Rav})\big).$$

3. Tamely ramified Local Langlands

The next easiest instance of local Langlands is the *tamely ramified* local Langlands conjecture. Here K is the *Iwahori subgroup* I, the pre-image of a Borel subgroup $B(k)$ under the reduction map $G(\mathcal{O}) \to G(k)$. Again such representations are controlled by the Iwahori-Hecke algebra $\mathbb{C}[I\backslash G(F)/I]$. In fact, this algebra only depends on the size of the residue field q, and there is a $\mathbb{C}[v^{\pm 1}]$-algebra $\mathcal{H}(\widehat{W})$ such that

$$\mathbb{C}[I\backslash G(F)/I] \simeq \mathcal{H}(\widehat{W})_{v=q}.$$

Kazhdan and Lusztig [KL87] succeeded in describing $\mathcal{H}(\widehat{W})$ in terms of the Langlands dual group $\check{G}$, proving tamely ramified local Langlands. To state their result, we take a quick detour to the geometry of the nilpotent cone.

Definition 3.1. Let $\mathcal{N} \subset \mathfrak{g}$ be the subscheme of nilpotent elements. As a scheme, it is the pre-image of 0 under the quotient $\mathfrak{g} \to \mathfrak{g}/\!/G \simeq \mathfrak{t}/\!/W$.

Example 3.2. When $\mathfrak{g} = \mathfrak{sl}_2$,

$$\mathcal{N} = \Big\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} : x^2 + yz = 0 \Big\} \subset \mathfrak{sl}_2.$$

As can be seen already from the example of $\mathfrak{g} = \mathfrak{sl}_2$ the scheme $\mathcal{N}$ is always singular (unless of course $\mathfrak{g}$ is abelian). However, it has a very nice resolution of singularities, known as the *Springer resolution*. Let $\widetilde{\mathcal{N}} = T^*(G/B) = G \times^B \mathfrak{n}$ which maps to $\mathcal{N}$ via

$$G \times^B \mathfrak{n} \to \mathcal{N} : (g, x) \mapsto \mathrm{ad}(g)x.$$

Example 3.3. When $\mathfrak{g} = \mathfrak{sl}_2$, the Springer resolution $T^*\mathbb{P}^1 \to \mathcal{N}$ is simply the blow-up of $\mathcal{N}$ at 0.

We will consider the *Steinberg variety*, $\mathrm{St}_G := \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$.

Proposition 3.4 (Kazhdan-Lusztig '87 [KL87])**.** *There is a ring isomorphism*

$$\mathcal{H}(\widehat{W}) \simeq K^{\check{G}\times\mathbb{G}_m}(\mathrm{St}_{\check{G}}) \tag{3.1}$$

where $K^{\check{G}\times\mathbb{G}_m}(\mathrm{St})$ denotes the $\check{G}\times\mathbb{G}_m$-equivariant K-theory of $\mathrm{St}_{\check{G}}$ (the $\mathbb{G}_m$ acts by dilation).

As a consequence, they deduce the Deligne-Langlands correspondence.

Corollary 3.2. *There is a bijection between:*

- *irreducible representations of $G(F)$ with an I-fixed vector; and*
- *a tuple (s, u, χ) of a semisimple element $s \in \check{G}$ and a unipotent element $u \in \check{G}$ such that $sus^{-1} = u^q$ and an irreducible representation χ of the centralizer $\pi_0 \mathrm{Z}_{\check{G}}(s, u)$, all up to $\check{G}$-conjugation.*

Again we hope to categorify the isomorphism (3.1), so the left-hand side naturally categorifies to $\mathrm{Perv}(I\backslash LG/I)$ where I is the pre-image of $B \subset G$ under $L^+G \to G$ and the right-hand side naturally categorifies to the category of $\check{G}$-equivariant coherent sheaves $\mathrm{Coh}^{\check{G}}(\mathrm{St}_{\check{G}})$. We thus expect, in analogy with Theorem 2.2, an equivalence

$$\mathrm{Perv}(I\backslash LG/I) \simeq \mathrm{Coh}^{\check{G}}(\mathrm{St}_{\check{G}}).$$

This turns out to be too good to be true. However, such an equivalence *does* hold if we work on the derived level.

Theorem 3.5 (Bezrukavnikov [Bez16])**.** *There is a monoidal equivalence of triangulated categories (or ∞-categories)*

$$\mathcal{D}_I := D^b_{\mathrm{cons}}(I\backslash LG/I) \simeq D^b_{\mathrm{coh}}(\check{G}\backslash(\widetilde{\mathcal{N}} \times^{\mathbb{L}}_{\check{\mathfrak{g}}} \widetilde{\mathcal{N}})). \tag{3.3}$$

Here, it is important that $\widetilde{\mathcal{N}} \times^{\mathbb{L}}_{\check{\mathfrak{g}}} \widetilde{\mathcal{N}}$ is considered as a *derived* scheme.

Remark 3.6. Analogous to Remark 2.6, the stack $\check{G}\backslash(\widetilde{\mathcal{N}} \times^{\mathbb{L}}_{\check{\mathfrak{g}}} \widetilde{\mathcal{N}}) \simeq \check{N}/\check{B} \times_{\check{G}/\check{G}} \check{N}/\check{B}$ also admits a description in terms of local systems. Indeed, the quotient $\check{N}/\check{B} = \check{B}/\check{B} \times_{\check{T}/\check{T}} \mathrm{pt}/\check{T}$ is the stack classifying a $\check{T}$-local system $\mathcal{E}$ on D and a $\check{B}$-local system $\mathcal{F}$ on $D^\times$ with an isomorphism $\mathcal{E}|_{D^\times} \simeq \mathcal{F} \times^{\check{B}} \check{T}$.

The prove (3.3), we first categorify a certain module of $\mathcal{H}(\widehat{W})$. Classically, an important observation in the representation theory of p-adic groups is that many irreducible representations of $G(F)$ are realized as subrepresentations of the *Whittaker module* $\mathrm{cind}_{N(F)}^{G(F)} \psi$ where $\psi\colon N(F) \to \mathbb{C}^\times$ is a non-degenerate character. In terms of representations of $\mathcal{H}(\widehat{W})_{v=q}$, let $I_0 \subset G(\mathcal{O})$ be the pre-image of $N(k) \subset G(k)$ under the reduction map $G(\mathcal{O}) \to G(k)$. Take a non-degenerate character $\psi\colon N(k) \to \mathbb{C}^\times$ and consider the space

$$\mathbb{C}[(I_0,\psi)\backslash G(F)/I] = \{f\colon G(F) \to \mathbb{C} : f(u_1 g u_2) = \psi(u_1) f(g) \text{ for } u_1 \in I_0, u_2 \in I\}$$

which is a $\mathcal{H}(\widehat{W})_{v=q}$-module by convolution. Then there turns out to be an isomorphism

$$\mathbb{C}[(I_0,\psi)\backslash G(F)/I] \simeq \mathbb{C}[X^*(\check{T})]. \tag{3.4}$$

We hope to categorify the isomorphism (3.4). In the categorical analog, we let I_0 be the pre-image of $N \subset G$ under the reduction map $L^+G \to G$, and fix a non-degenerate character $\psi\colon I_0 \to \mathbb{G}_a$. We may then take the pre-image of the Artin-Schrier sheaf AS and consider the *Iwahori-Whittaker* category

$$\mathcal{D}_{IW} := D_{\mathrm{cons}}((I_0, \psi^*\mathrm{AS})\backslash LG/I).$$

Theorem 3.7 (Arkhipov–Bezrukavnikov [AB09])**.** *There is an equivalence*

$$\mathcal{D}_{IW} \simeq D^b_{\mathrm{coh}}(\check{G}\backslash\widetilde{\mathcal{N}}) = D^b_{\mathrm{coh}}(\check{B}\backslash\check{\mathfrak{n}}).$$

The goal of the seminar is to cover the proofs of Theorem 2.2 and Theorem 3.7. If time and participants permit, we will discuss Theorem 3.5 in a suitable follow-up format (workshop or seminar).

References

[AB09] Sergey Arkhipov and Roman Bezrukavnikov, *Perverse sheaves on affine flags and Langlands dual group*, Israel J. Math. **170** (2009), 135–183, With an appendix by Bezrukavnikov and Ivan Mirković. MR 2506322

[Bez16] Roman Bezrukavnikov, *On two geometric realizations of an affine Hecke algebra*, Publ. Math. Inst. Hautes Études Sci. **123** (2016), 1–67. MR 3502096

[BF08] Roman Bezrukavnikov and Michael Finkelberg, *Equivariant Satake category and Kostant-Whittaker reduction*, Mosc. Math. J. **8** (2008), no. 1, 39–72, 183. MR 2422266

[Gin00] Victor Ginzburg, *Perverse sheaves on a Loop group and Langlands' duality*, 2000.

[KL87] David Kazhdan and George Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), no. 1, 153–215. MR 862716

[MV07] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143. MR 2342692

M.I.T., 77 Massachusetts Avenue, Cambridge, MA, USA

Email address: kjsuzuki@mit.edu