Notes for Geometric Representation Theory Seminar

March 10, 2026

Throughout this lecture, we fix a connected reductive algebraic group G over $\mathbf{C}$. Set $\mathcal{K} = \mathbf{C}((t))$ and $\mathcal{O} = \mathbf{C}[[t]]$, and let $\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$ be the affine Grassmannian of G. This has the structure of an Ind-scheme over $\mathbf{C}$, but for our purposes we will largely view it merely as a topological space. Fix a coefficient field k (of any characteristic) and let $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ denote the Satake category of $G(\mathcal{O})$-equivariant perverse sheaves (of k-vector spaces) on Gr_G. The category $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ has a monoidal structure given by the convolution product $(\mathscr{E}, \mathscr{F}) \mapsto \mathscr{E} \star \mathscr{F}$.

In the last lecture, we saw that this convolution product has an alternative description (given by "fusion"). This alternative description yields canonical isomorphisms $\mathscr{E} \star \mathscr{F} \simeq \mathscr{F} \star \mathscr{E}$, making $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ into a symmetric monoidal category. Our goal over the next few lectures is to discuss the following:

Proto-Theorem 1 (Geometric Satake Isomorphism)**.** There is a symmetric monoidal equivalence of $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ with the category of representations of the Langlands dual group $\check{G}$ (regarded as a reductive algebraic group over k).

Beware that Proto-Theorem 1 is false as stated. In the last lecture, we defined a "fiber functor"

$$H : \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \to \{k\text{-vector spaces}\} \qquad H(\mathscr{E}) := \mathrm{H}^*(\mathrm{Gr}_G, \mathscr{E})$$

and showed that it is compatible with convolution: for equivariant perverse sheaves $\mathscr{E}$ and $\mathscr{F}$, there is a canonical vector space isomorphism

$$H(\mathscr{E} \star \mathscr{F}) \simeq H(\mathscr{E}) \otimes_k H(\mathscr{F}).$$

This construction endows H with the structure of a monoidal functor. However, it is not a *symmetric* monoidal functor: that is, it is not compatible with the commutativity constraint of the previous lecture. To see this, we note that H factors as a composition

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \xrightarrow{H^*} \{\text{Graded } k\text{-vector spaces}\} \xrightarrow{\bigoplus} \{k\text{-vector spaces}\}.$$

The category in the middle has two *different* symmetric monoidal structures:

(a) Given graded vector spaces V^* and W^*, there is a canonical isomorphism $\alpha : V^* \otimes W^* \simeq W^* \otimes V^*$, given by the direct sum of the tautological identifications $\alpha^{ij} : V^i \otimes_k W^j \simeq W^j \otimes_k V^i$.

(b) There is another isomorphism of graded vector spaces $\beta : V^* \otimes W^* \simeq W^* \otimes V^*$ given by the *Koszul sign rule*: it is given by the direct sum of the isomorphisms $\beta^{ij} := (-1)^{ij} \alpha^{ij}$.

Here the forgetful functor

$$\{\text{Graded } k\text{-vector spaces}\} \to \{k\text{-vector spaces}\} \qquad V^* \mapsto \bigoplus V^n$$

is symmetric monoidal with respect to the first convention, but the functor

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \to \{\text{Graded } k\text{-vector spaces}\} \qquad \mathscr{E} \mapsto H^*(\mathrm{Gr}_G, \mathscr{E})$$

is symmetric monoidal only if we use the Koszul sign rule.

From this, we can already see that Theorem 1 cannot possibly be true. Recall that, if $\mathcal{C}$ is a symmetric monoidal category with unit object $\mathbf{1}$, then every dualizable object $V \in \mathcal{C}$ has a well-defined *dimension* $\dim(V) \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$, given by the trace of the identity map $\mathrm{id}_V : V \to V$. This specializes to several well-known invariants:

Example 1. Let $\mathcal{C}$ be the category of vector spaces over k. Then an object $V \in \mathcal{C}$ is dualizable if and only if it is finite-dimensional in the usual sense. In this case, the dimension $\dim(V)$ defined above coincides with its usual dimension (regarded as an element of the field k).

Example 2. Let $\mathcal{C}$ be the category of representations of an algebraic group $\check{G}$ over k. The forgetful functor $\mathcal{C} \to \{k\text{-vector spaces}\}$ is symmetric monoidal, and therefore preserves the formation of dimensions. It follows that the dimension of a dualizable object $V \in \mathcal{C}$ coincides with the usual dimension of its underlying vector space, again regarded as an element of the field k.

Example 3. Let $\mathcal{C}$ be the category of graded vector spaces over k, endowed with the symmetric monoidal structure given by the Koszul sign rule. In this case, the dimension of an object $V^* \in \mathcal{C}$ (in the sense defined above) is the *superdimension* of V as a graded vector space: that is, the alternating sum $\sum_{n \in \mathbf{Z}} (-1)^n \mathrm{dim}_k(V^n)$.

Example 4. Set $G = \mathrm{GL}_2$. As a set, the affine Grassmannian Gr_G can be identified with the collection of $\mathbf{C}[[t]]$-lattices $L \subseteq \mathbf{C}((t))^2$. Let $X \subseteq \mathrm{Gr}_G$ be the locus consisting of lattices L satisfying $t\,\mathbf{C}[[t]]^2 \subsetneq L \subsetneq C[[t]]^2$. Then X is a (closed) orbit for the action of $G(\mathcal{O})$ on Gr_X, isomorphic (as a variety) to the projective line $\mathbf{P}^1$. In particular, the constant sheaf

$\underline{k}_X[1]$ is an equivariant perverse sheaf on Gr_G, which we can identify with an object $\mathscr{E}$ of the Satake category. We then have

$$\mathrm{H}^*(\mathrm{Gr}_G, \mathscr{E}) = \mathrm{H}^{*+1}(\mathbf{P}^1, k) = \begin{cases} k & \text{if } * \in \{-1, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Since the functor $\mathscr{E} \mapsto \mathrm{H}^*(\mathrm{Gr}_G, \mathscr{E})$ is symmetric monoidal (using the Koszul sign rule), it follows that the dimension of $\mathscr{E}$ as an object of the Satake category coincides with the superdimension of the graded vector space $\mathrm{H}^*(\mathrm{Gr}_G, \mathscr{E})$, which is equal to -2.

In particular, if k has characteristic zero, then $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ cannot be equivalent (as a symmetric monoidal category) to the category of representations of an algebraic group.

To address the point raised above, it will be necessary either to restrict the class of groups under consideration (for example, by requiring G to be simply connected) or to modify the commutativity constraint on the Satake category. This is a concern for a future lecture. Our goal for today is to prove the following:

Theorem 5. *The functor* $H : \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \to \{k\text{-vector spaces}\}$ *is exact and faithful.*

Remark 6. Suppose we are given a short exact sequence

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$$

in the Satake category. We then have a long exact sequence of cohomology groups

$$\cdots \to \mathrm{H}^{n-1}(\mathrm{Gr}_G, \mathscr{E}'') \xrightarrow{\partial} \mathrm{H}^n(\mathrm{Gr}_G, \mathscr{E}') \to \mathrm{H}^n(\mathrm{Gr}_G, \mathscr{E}) \to \mathrm{H}^n(\mathrm{Gr}_G, \mathscr{E}'') \xrightarrow{\partial} \mathrm{H}^{n+1}(\mathrm{Gr}_G, \mathscr{E}') \to \cdots$$

The content of the exactness assertion of Theorem 5 is that the boundary maps appearing in this sequence are zero.

To prove Theorem 5, we will introduce a refinement of the functor H, which can be regarded as a categorified analogue of the *Satake transform* from an earlier lecture. Before getting into the details, let us embark on brief digression on the Satake transform. Here we temporarily return to the "classical" picture where $\mathcal{K}$ is a nonarchimedean local field (for example, $\mathbf{F}_q((t))$) and $\mathcal{O}$ is its ring of integers. Recall that a (left) action of $G(\mathcal{K})$ on a set X (or abelian group, or k-vector space) is *smooth* if the stabilizer of any element $x \in X$ is an open subgroup of $G(\mathcal{K})$.

Example 7. Let H be an open subgroup of $G(\mathcal{K})$. Then the set of right cosets $G(\mathcal{K})/H$ carries a smooth (left) action of $G(\mathcal{K})$. We will be particularly interested in the case $H = G(\mathcal{O})$.

Example 8. Let X be a set with a smooth left action of $G(\mathcal{K})$, and let $V = k[X]$ be the free k-vector space generated by X. Then $V = k[X]$ inherits a smooth left action of $G(\mathcal{K})$. We will be particularly interested in the special case $X = G(\mathcal{K})/G(\mathcal{O})$. Then V can be identified with the collection of finitely supported k-valued functions on $G(\mathcal{K})/G(\mathcal{O})$, or equivalently with the collection of compactly supported k-valued functions on $G(\mathcal{K})$ which are invariant under right translation by $G(\mathcal{O})$ (and therefore locally constant).

The representation V is characterized by a universal property: if W is a k-vector space with an action of $G(\mathcal{K})$, we have a canonical bijection

$$\mathrm{Hom}(V, W) = \mathrm{Hom}_{G(\mathcal{K})}(G(\mathcal{K})/G(\mathcal{O}), W) = W^{G(\mathcal{O})}.$$

Here the Hom on the left hand side is computed in the category of (smooth) k-linear representations of $G(\mathcal{K})$, and the right hand side is the space of $G(\mathcal{O})$-fixed points W.

Construction 9 (The Hecke Algebra)**.** Set $V = k[G(\mathcal{K})/G(\mathcal{O})]$ as above. We let $\mathcal{H}_G$ denote the endomorphism ring $\mathrm{End}(V)$, computed in the category of (smooth) k-linear representations of G. We refer to $\mathcal{H}_G$ as the *Hecke algebra* of G. Invoking the universal property of V, we obtain

$$\mathcal{H}_G = \mathrm{Hom}(V, V) = V^{G(\mathcal{O})} = \{\text{Compactly supported } G(\mathcal{O})\text{-biinvariant functions on } G(\mathcal{K})\}.$$

By construction, the Hecke algebra $\mathcal{H}_G = \mathrm{Hom}(V, V)$ acts on the space $\mathrm{Hom}(V, W) = W^{G(\mathcal{O})}$ for every (smooth) k-linear representation of $G(\mathcal{K})$ (this is *a priori* a right action, but we saw earlier that the Hecke algebra is commutative so the distinction is not so important).

Remark 10 (Relationship with the Satake Category)**.** Let us now suppose that $\mathcal{K} = \mathbf{F}_q((t))$ is an equicharacteristic local field, whose residue field $\mathbf{F}_q$ is finite of characteristic $p > 0$. Let us (temporarily) regard the affine Grassmannian Gr_G as an Ind-scheme over $\mathbf{F}_q$. We can then identify the quotient $G(\mathcal{K})/G(\mathcal{O})$ with the set of $\mathbf{F}_q$-rational points of Gr_G (this identification is not completely obvious: it uses Lang's theorem, which guarantees that every G-bundle over $\mathrm{Spec}(\mathbf{F}_q)$ is trivial). Take $k = \mathbf{Q}_\ell$ for some $\ell \neq p$. We then have Grothendieck's *function-sheaf correspondence*, which associates to every ℓ-adic perverse sheaf $\mathscr{E}$ on Gr_G a finitely supported $\mathbf{Q}_\ell$-valued function $T_{\mathscr{E}}$ on $G(\mathcal{K})/G(\mathcal{O}) = \mathrm{Gr}_G(\mathbf{F}_q)$, given concretely by the formula

$$T_{\mathscr{E}}(x) := \sum_n (-1)^n \mathrm{Tr}(\varphi : \mathrm{H}^n(\mathscr{E}_{\overline{x}}) \to \mathrm{H}^n(\mathscr{E}_{\overline{x}}).$$

If $\mathscr{E}$ is equivariant, then the function $T_{\mathscr{E}}$ is invariant under translation by $G(\mathcal{O})$, and can therefore be regarded as an element of the Hecke algebra $\mathcal{H}_G$ of Construction 9. This construction determines an algebra homomorphism

$$T : K_0(\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)) \to \mathcal{H}_G\,.$$

The compatibility of convolution with multiplication in the Hecke algebra is a consequence of the Grothendieck-Lefschetz trace formula.

The algebra homomorphism T of Remark **??** fits into a commutative diagram

$$\begin{tikzcd}
K_0(\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)) \arrow[r, "T"] \arrow[d] & \mathcal{H}_G \arrow[d, "\int"] \\
K_0(\{\mathbf{Q}_\ell\text{-Vector spaces}\}) \arrow[r] & \mathbf{Q}_\ell,
\end{tikzcd}$$

where the right vertical map carries a $G(\mathcal{O})$-bi-equivariant function f on $G(\mathcal{K})$ to its integral $\int f d\mu$, where μ is the Haar measure on $G(\mathcal{K})$ normalized so that $G(\mathcal{O})$ has measure 1 (the commutativity of the diagram is again a consequence of the Grothendieck-Lefschetz trace formula, now for the affine Grassmannian Gr_G).

Beware that the vertical maps in this diagram are very destructive: for example, they are not injective. It will therefore be useful to consider a more refined construction which loses less information. In what follows, we fix a split maximal torus $T \subseteq G$. Let $B \subseteq G$ be a Borel subgroup containing T, and let $U \subseteq B$ be its unipotent radical. We can then consider the following constructions:

- Every (smooth) representation of $G(\mathcal{K})$ can be regarded as a (smooth) representation of $B(\mathcal{K})$, by neglect of structure.

- If V is a smooth representation of $B(\mathcal{K})$, then the $U(\mathcal{K})$-coinvariants

  $$V_{U(\mathcal{K})} := V/\{v - g(v) : v \in V, g \in U(k)\}$$

  inherits a (smooth) action of the quotient group $T(\mathcal{K}) = B(\mathcal{K})/U(\mathcal{K})$.

Composing these constructions, we obtain a functor

$$\{\text{Smooth } G(\mathcal{K})\text{-representations}\} \to \{\text{Smooth } T(\mathcal{K})\text{-representations}\} \qquad V \mapsto V_{U(K)}.$$

Let us study what happens for the representation $V = k[G(\mathcal{K})/G(\mathcal{O})]$ defined above.

Remark 11. The canonical map $B(\mathcal{K})/B(\mathcal{O}) \to G(\mathcal{K})/G(\mathcal{O})$ is bijective. Equivalently, the map $G(\mathcal{O})/B(\mathcal{O}) \to G(\mathcal{K})/B(\mathcal{K})$ is bijective. To prove this, let us consider the diagram

$$\begin{tikzcd}
G(\mathcal{O})/B(\mathcal{O}) \arrow[r] \arrow[d] & G(\mathcal{K})/B(\mathcal{K}) \arrow[d] \\
X(\mathcal{O}) \arrow[r] & X(\mathcal{K})
\end{tikzcd}$$

where X denotes the flag variety G/B. The lower horizontal map is a bijection by the valuative criterion of properness, and the vertical maps are injective. Consequently, to prove that the upper horizontal map is a bijection, it will suffice to prove the surjectivity of the map on the left. This follows from the observation that every B-bundle on $\mathrm{Spec}(\mathcal{O})$ is trivial.

It follows that, as a representation of $B(\mathcal{K})$, we can identify $k[G(\mathcal{K})/G(\mathcal{O})]$ with $k[B(\mathcal{K})/B(\mathcal{O})]$.

Remark 12. We have a (split) exact sequence of algebraic groups

$$0 \to U \to B \to T \to 0,$$

which induces a surjection $B(\mathcal{K})/B(\mathcal{O}) \twoheadrightarrow T(\mathcal{K})/T(\mathcal{O})$. We claim that this surjection exhibits the right hand side as the quotient of $B(\mathcal{K})/B(\mathcal{O})$ by the (left) action of $U(\mathcal{K})$. More concretely, suppose we are given a pair of elements $b, b' \in B(\mathcal{K})$ having the same image in $T(\mathcal{K})/T(\mathcal{O})$. Then we can write b and b' (uniquely) as products $b = ut$ and $b' = u't'$, for $u, u' \in U(\mathcal{K})$ and $t, t' \in T(\mathcal{K})$. Then $b' = (u'u^{-1}) \cdot b \cdot (t^{-1}t')$, where the first factor lies in $U(\mathcal{K})$ and the third factor lies in $T(\mathcal{O})$.

Taking free vector spaces on both sides, we obtain an isomorphism

$$k[B(\mathcal{K})/B(\mathcal{O})]_{U(\mathcal{K})} \simeq k[T(\mathcal{K})/T(\mathcal{O})].$$

Combining the preceding analyses, we see that the functor

$$\{\text{Smooth } G(\mathcal{K})\text{-representations}\} \to \{\text{Smooth } T(\mathcal{K})\text{-representations}\} V \mapsto V_{U(\mathcal{K})}$$

carries $k[G(\mathcal{K})/G(\mathcal{O})]$ to $k[T(\mathcal{K})/T(\mathcal{O})]$, and therefore induces a map of endomorphism rings

$$S^{\mathrm{un}} : \mathcal{H}_G \to \mathcal{H}_T$$

which we will refer to as the *unnormalized Satake transform.* Concretely, if we think about $\mathcal{H}_G$ and $\mathcal{H}_T$ as consisting of bi-invariant k-valued functions on $G(\mathcal{K})$ and $T(\mathcal{K})$, respectively, then it is given concretely by the formula

$$S^{\mathrm{un}}(f)(t) = \int_{u \in U(\mathcal{K})} f(tu) d\mu,$$

where μ denotes the Haar measure on $U(\mathcal{K})$ normalized so that $U(\mathcal{O})$ has volume 1.

Warning 13. Let us equip $\mathcal{K}$ with its normalized absolute value $|\bullet|_{\mathcal{K}}$ (so that a uniformizer $\pi \in \mathcal{K}$ has absolute value $1/q$). Let $(2\rho) : T \to \mathbf{G}_m$ be the character of G given by the sum of the positive roots (if G is simply connected, then this character has a square root $\rho : T \to \mathbf{G}_m$ given by the sum of the fundamental weights). Then conjugation by an element $t \in T(\mathcal{K})$ determines an automorphism of $U(\mathcal{K})$ which scales the Haar measure μ by a factor of $|(2\rho)(t)|_{\mathcal{K}}^{-1}$ (the appearance of the inverse here depends on a choice of convention). It follows that we can rewrite the unnormalized Satake transform as

$$S^{\mathrm{un}}(f)(t) = |(2\rho)(t)|_{\mathcal{K}}^{-1} \int_{u \in U(\mathcal{K})} f(ut) d\mu.$$

Note that this differs from the Satake transform from the second lecture, given by the formula

$$S(f)(t) = |\rho(t)|_{\mathcal{K}}^{-1} \int_{u \in U(\mathcal{K})} f(ut) d\mu = |\rho(t)|_{\mathcal{K}} \int_{u \in U(\mathcal{K})} f(tu) d\mu.$$

Beware that this generally requires us to assume that k contains a square root of q (or to restrict our attention to groups G where ρ is a well-defined character, such as simply-connected groups).

Our goal now is to study a counterpart of the (unnormalized) Satake transform in the geometric setting.

Construction 14 (The Constant Term)**.** The diagram of algebraic groups $G \supset B \twoheadrightarrow T$ determines a diagram of affine Grassmannians

$$\mathrm{Gr}_G \xleftarrow{f} \mathrm{Gr}_B \xrightarrow{g} \mathrm{Gr}_T .$$

On equivariant derived categories of k-valued sheaves, we obtain a functor

$$D_{L^+G}(\mathrm{Gr}_G) \to D_{L^+(T)}(\mathrm{Gr}_T) \qquad \mathscr{E} \mapsto g_! f^* \mathscr{E}$$

which we refer to as the *constant term functor*.

If we work over the finite field $\mathbf{F}_q$ (and use ℓ-adic sheaves), this construction is compatible (under the function-sheaf correspondence) with the unnormalized Satake transform $\mathcal{H}_G \to \mathcal{H}_T$ described above.

Let us now return to our original "geometric" setting (we regard the affine Grassmannian as defined over the complex numbers, with $\mathcal{K} = \mathbf{C}((t))$ and $\mathcal{O} = \mathbf{C}[[t]]$), and study the preceding construction in more detail. Recall that every cocharacter $\mu : \mathbf{G}_m \to T$ determines an element $t^\mu \in T(\mathcal{K}) \subset G(\mathcal{K})$. Moreover, the construction $\mu \mapsto t^\mu$ induces a homeomorphism from the cocharacter lattice $X_*(T)$ to the affine Grassmannian $\mathrm{Gr}_T = T(\mathcal{K})/T(\mathcal{O})$. We will write x_μ for the point of Gr_G given by the coset $t^\mu G(\mathcal{O})$.

Arguing as in Remark 11, we see that the map $\mathrm{Gr}_B \to \mathrm{Gr}_G$ is a bijection at the level of points (this is even easier in the geometric setting: every B-bundle on $\mathrm{Spec}(\mathbf{C})$ is automatically trivial). However, it is not a homeomorphism of topological spaces. For example, if G is simply connected, then Gr_G is connected. However, Gr_B breaks up as a disjoint union of components $\mathrm{Gr}_B^\mu = \mathrm{Gr}_B \times_{\mathrm{Gr}_T} \{\mu\}$. More concretely, Gr_B^μ can be written as the quotient

$$\mathrm{Gr}_B^\mu = (U(\mathcal{K}) \cdot t^\mu T(\mathcal{O}))/(U(\mathcal{O}) \cdot T(\mathcal{O})) = (U(\mathcal{K}) t^\mu)/U(\mathcal{O}) \simeq U(\mathcal{K})/(t^\mu U(\mathcal{O}) t^{-\mu}).$$

From this description one sees that Gr_B^{μ} looks like an affine space of infinite dimension: more precisely, it can be realized as a directed colimit of finite-dimensional affine spaces over $\mathbf{C}$. The image Gr_B^{μ} in the affine Grassmannian Gr_G is a subset $S_\mu \subseteq \mathrm{Gr}_G$ which we refer to as a *semi-infinite orbit.* Concretely, S_μ is the $U(\mathcal{K})$-orbit of Gr_G which contains the point $x_\mu := t^{\mu} G(\mathcal{O})$. From the above, we see that Gr_G decomposes as a disjoint union of semi-infinite orbits S_μ as the cocharacter μ varies.

Fact 15. *For every cocharacter μ, the closure of the semi-infinite orbit S_μ is the union $\bigcup_{\nu \leq \mu} S_\nu$. Recall that we write $\nu \leq \mu$ when $\nu - \mu$ can be realized as a sum of positive coroots.*

Example 16. In the case $G = \mathrm{SL}_2$, we have an order-preserving identification $\mathbf{Z} \simeq X_*(T)$, carrying 1 to the standard cocharacter

$$\mathbf{G}_m \to \mathrm{SL}_2 \qquad u \mapsto \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}.$$

Recall that we can identify Gr_G with the collection of $\mathbf{C}[[t]]$-lattices $L \subseteq \mathbf{C}((t))^2$ which are "of unit volume", meaning that the quotient $\mathbf{C}[[t]]/t^d L$ has complex dimension $2d$ for $d \gg 0$. Let e and e' be the standard basis of $\mathbf{C}((t))^2$, and let μ be a cocharacter corresponding to an integer m. Then:

- A lattice L belongs to the closure of the semi-infinite orbit S_μ if and only if it contains $t^m e$.
- A lattice L belongs to the semi-infinite orbit S_μ itself if and only if it contains $t^m e$ and does not contain $t^d e$ for $d < m$.

In the latter case, the "unit volume" condition guarantees that L is spanned by the vectors $t^m e$ and $f(t)e + t^{-m}e'$, for some element $f(t) \in \mathbf{C}((t))$ which is well-defined modulo t^m. This gives an identification of the semi-infinite orbit S_μ with the quotient $\mathbf{C}((t))/t^m\,\mathbf{C}[[t]]$.

We can now rephrase Construction 14 more concretely: given an equivariant sheaf $\mathscr{E}$ on the affine Grassmanian Gr_G, its constant term is the family of complexes

$$\{\mathrm{R}\Gamma_c(S_\mu, \mathscr{E}|_{S_\mu})\}_{\mu \in X_*(T)},$$

which we regard as an $X_*(T)$-graded object of the derived category $D(k)$. We will be interested in the special case where $\mathscr{E}$ is perverse, in which case we can be more precise:

Theorem 17. *Let $\mathscr{E}$ be a $G(\mathcal{O})$-equivariant perverse sheaf on Gr_G and let μ be a cocharacter of T. Then the compactly supported cohomology groups*

$$\mathrm{H}_c^*(S_\mu, \mathscr{E}|_{S_\mu})$$

are concentrated in the single degree $* = \langle 2\rho, \mu \rangle$.

Corollary 18. *Let μ be a cocharacter of T and set $d = \langle 2\rho, \mu\rangle$. Then the functor*

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \to \{\text{Vector spaces over } k\} \qquad \mathscr{E} \mapsto \mathrm{H}^d_c(S_\mu, \mathscr{E}|_{S_\mu})$$

is exact.

Corollary 19. *For $\mathscr{E} \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$, we have a canonical isomorphism*

$$\mathrm{H}^*(\mathrm{Gr}_G, \mathscr{E}) \simeq \bigoplus_{\mu \in X_*(T)} \mathrm{H}^{\langle 2\rho,\mu\rangle}_c(S_\mu, \mathscr{E}|_{S_\mu}).$$

Note that the degrees appearing in Corollary 19 can be both odd and even, since the character $2\rho \in X^*(T)$ is not necessarily divisible by 2.

Proof. For every integer n, let $\mathrm{Gr}_G^{\leq n}$ denote the union of the semi-infinite orbits S_μ satisfying $\langle 2\rho, \mu\rangle \leq n$. It follows from Fact 15 that each $\mathrm{Gr}_G^{\leq n}$ is closed in Gr_G, and that each $\mathrm{Gr}_G^{\leq n} \setminus \mathrm{Gr}_G^{\leq n-1}$ is a disjoint union of those semi-infinite orbits S_μ satisfying $\langle 2\rho, \mu\rangle = n$. This determines a (finite) decreasing filtration on the complex $\mathrm{R}\Gamma(\mathrm{Gr}_G, \mathscr{E})$ where

$$\mathrm{gr}^n \,\mathrm{R}\Gamma(\mathrm{Gr}_G, \mathscr{E}) \simeq \bigoplus_{\langle 2\rho,\mu\rangle = n} \mathrm{R}\Gamma_c(S_\mu, \mathscr{E}|_{S_\mu})$$

is concentrated in cohomological degree n. It follows that $\mathrm{H}^*(\mathrm{Gr}_G, \mathscr{E})$ can be computed as the cohomology of a chain complex

$$\cdots \to \bigoplus_{\langle 2\rho,\mu\rangle = n-1} H^{n-1}(S_\mu, \mathscr{E}|_{S_\mu}) \xrightarrow{\partial} \bigoplus_{\langle 2\rho,\mu\rangle = n} H^{n}(S_\mu, \mathscr{E}|_{S_\mu}) \to \cdots,$$

and it will suffice to show that all of the differentials are zero. For this, we can assume that $\mathscr{E}$ is supported on a single connected component of the affine Grassmannian Gr_G. We now observe that if μ and ν are cocharacters for which S_μ and S_ν belong to the same connected component of Gr_G, then μ and ν differ by a sum of coroots (Fact 15) so that $\langle 2\rho, \mu\rangle$ and $\langle 2\rho, \nu\rangle$ have the same parity. It follows that our chain complex is concentrated either in even or odd degrees, so the differential is automatically zero. □

The proof of Theorem 17 breaks naturally into two parts: given an equivariant perverse sheaf $\mathscr{E}$ on Gr_G, we need both lower and upper bounds for the degrees in which the groups $\mathrm{H}^*_c(S_\lambda, \mathscr{E}|_{S_\mu})$ are concentrated. Let us begin with the upper bounds. Recall that a constructible complex $\mathscr{E}$ on Gr_G is *semiperverse* if, for every integer d, the locus $\{x \in \mathrm{Gr}_G : \mathrm{H}^{-d}(\mathscr{E}_x) \neq 0\}$ has dimension $\leq d$ (and therefore vanishes for $d < 0$). Every perverse sheaf is semiperverse (by definition). Half of Theorem 17 is therefore contained in the following:

Proposition 20. *Let $\mathscr{E}$ be a $G(\mathcal{O})$-equivariant constructible complex on* Gr_G *which is semiperverse and let μ be a cocharacter of T. Then the compactly supported cohomology groups*

$$\mathrm{H}^*_c(S_\mu, \mathscr{E}|_{S_\mu})$$

are concentrated in degrees $\leq \langle 2\rho, \mu \rangle$.

Note that $\mathscr{E}$ is semiperverse if and only if it can be realized as an iterated extension of sheaves of the form $\mathscr{L}[d]$, where $\mathscr{L}$ is a trivial local system on some $G(\mathcal{O})$-orbit Gr^ν_G having dimension $\leq d$, regarded as a sheaf on Gr_G via extension-by-zero. Recall that the orbit Gr^ν_G has dimension $\langle 2\rho, \nu \rangle$. In this case, we have an isomorphism

$$\mathrm{H}^*_c(S_\mu, \mathscr{E}|_{S_\lambda}) \simeq \mathrm{H}^{*+d}_c(S_\mu \cap \mathrm{Gr}^\nu_G, k),$$

which is concentrated in degrees $\leq 2\mathrm{dim}(S_\mu \cap \mathrm{Gr}^\nu_G) - d$ (and nonvanishing in the top degree). We can therefore restate Proposition 20 as follows:

Proposition 21. *Let μ and ν be cocharacters of T where μ is dominant. Then the intersection $S_\mu \cap \mathrm{Gr}^\nu_G$ has dimension $\leq \langle \rho, \mu + \nu \rangle$.*

In fact, one can be more precise: every connected component of the intersection $S_\mu \cap \mathrm{Gr}^\nu_G$ has dimension exactly equal to $\langle \rho, \mu + \nu \rangle$ (note that $\langle \rho, \mu + \nu \rangle$ might not be an integer: in this case, the intersection is empty because S_μ and Gr^ν_G belong to different connected components of Gr_G).

We will not give the proof of Proposition 21 here.

Example 22. Let $G = \mathrm{SL}_2$. As in Example 16, we can identify Gr_G with the collection of unit volume lattices $L \subseteq \mathbf{C}((t))^2 = \mathbf{C}((t))e \oplus \mathbf{C}((t))e'$. Let us identify μ and ν with the integers $m = \langle \rho, \mu \rangle$ and $n = \langle \rho, \nu \rangle$, where $n \geq 0$. In this case:

- The semi-infinite orbit S_μ corresponds to those lattices L which are generated by $t^m e$ and $f(t)e + t^{-m}e'$, for some $f(t) \in \mathbf{C}((t))$.
- The closed stratum $\overline{\mathrm{Gr}}^\nu_G$ corresponds to those lattices L which are contained in $t^{-n}\,\mathbf{C}[[t]]$.

If *both* of these conditions are satisfied, then $f(t)$ must be contained in $t^{-n}\,\mathbf{C}[[t]]$. Since $f(t)$ is only well-defined modulo $t^m\,\mathbf{C}[[t]]$, the dimension of the intersection is bounded by $m + n$.

Proof of Theorem 5. It follows immediately from Corollaries 19 and 18 that the functor

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \to \{k\text{-vector spaces}\} \qquad \mathscr{E} \mapsto \mathrm{H}^*(\mathrm{Gr}_G, \mathscr{E})$$

is exact. Let us show that it is faithful: that is, it does not annihilate any nonzero perverse sheaves $\mathscr{E}$. Without loss of generality, we may assume that $\mathscr{E}$ is irreducible, corresponding to the intersection homology sheaf of a closed stratum $\overline{\mathrm{Gr}}_G^{\nu}$ for some dominant cocharacter ν.

Let w be an element of the Weyl group $W = N(T)/T$, and let $\widetilde{w}$ be a representative of w in the group $G(\mathbf{C})$. Then

$$\widetilde{w}x_\nu = \widetilde{w}t^\nu G(\mathcal{O}) = t^{w(\nu)}\widetilde{w}G(\mathcal{O}) = x_{w(\nu)}.$$

It follows that the point $x_{w(\nu)} \in \mathrm{Gr}_G$ is contained in the orbit $\mathrm{Gr}_G^\nu = G(\mathcal{O})x_\mu$, and also in the semi-infinite orbit $S_{w(\nu)}$. Applying Proposition 21 in the case $\mu = w(\nu)$, we obtain

$$\dim(S_{w(\nu)} \cap \mathrm{Gr}_G^\nu) = \langle \rho, \nu + w(\nu)\rangle = \langle \rho + w^{-1}(\rho), \nu\rangle.$$

Recall that W contains a "longest element" w_{long}, characterized by the requirement that it carries positive roots to negative roots and therefore carries ρ to $-\rho$. Applying the preceding estimate in the case $\mu = w_{\mathrm{long}}(\nu)$, we conclude that the intersection $S_\mu \cap \mathrm{Gr}_G^\nu$ is zero-dimensional. A similar argument shows that S_μ does not intersect the boundary of Gr_G^ν, so that x_μ is an isolated point of $S_\mu \cap \overline{\mathrm{Gr}}_G^\nu$. It follows that the restriction $\mathscr{E}|_{S_\mu}$ contains a skyscraper sheaf as a direct summand. In particular, the compactly supported cohomology $\mathrm{H}_c^*(S_\mu, \mathscr{E}|_{S_\mu})$ does not vanish, so $\mathrm{H}^*(\mathrm{Gr}_G, \mathscr{E})$ also does not vanish (Corollary 19). $\square$

Before we complete the proof of Theorem 17, it will be useful to go on a brief digression, which gives an alternative description of the functor $\mathscr{E} \mapsto \mathrm{H}_c^*(S_\mu, \mathscr{E}|_{S_\mu})$. For each cocharacter μ, let $x_\mu : \mathrm{Spec}(\mathbf{C}) \hookrightarrow S_\mu \subset \mathrm{Gr}_G$ denote the point of the corresponding semi-infinite orbit given by the coset $t^\mu G(\mathcal{O})$. If $\mathscr{F}$ is a sheaf on S_μ, we write $\mathscr{F}_{x_\mu}^!$ for its costalk at the point x_μ (given by applying the functor $x_\mu^!$ to $\mathscr{F}$). More informally, this costalk computes sections of $\mathscr{F}$ which are supported at the point x_μ; in particular, there is a comparison map

$$\mathscr{F}_{x_\mu}^! \to \mathrm{R}\Gamma_c(S_\mu, \mathscr{F}).$$

Proposition 23. *For every equivariant constructible complex $\mathscr{E}$ on Gr_G, the preceding construction determines an isomorphism*

$$(\mathscr{E}|_{S_\mu})_{x_\mu}^! \to \mathrm{R}\Gamma_c(S_\mu, \mathscr{E}|_{S_\mu}).$$

Consequently, the constant term of Construction 14 can be identified with the collection of costalks

$$\{(\mathscr{E}|_{S_\mu})_{x_\mu}^!\}_{\mu \in X_*(T)}.$$

Let us explain the proof of Proposition 23. Along the way, we will develop some ideas which will be useful to complete the proof of Theorem 17. Note that the algebraic group G can be regarded as a subgroup of the loop group LG (consisting of "constant loops"), and therefore acts on the Grassmannian Gr_G. In particular, every cocharacter $\lambda : \mathbf{G}_m \to T \subset G$ determines an action of $\mathbf{G}_m$ on Gr_G, given concretely by the map

$$\mathbf{G}_m \times \mathrm{Gr}_G \to \mathrm{Gr}_G \qquad (s, gG(\mathcal{O})) = \lambda(s)gG(\mathcal{O}).$$

If μ is another cocharacter of T, then this restricts to a map

$$\mathbf{G}_m \times S_\mu \to \mathrm{Gr}_G \qquad (s, ut^\mu G(\mathcal{O})) \mapsto \lambda(s)ut^\mu G(\mathcal{O}) = (\lambda(s)u\lambda(s)^{-1})t^\mu G(\mathcal{O}).$$

(since $\lambda(s)$ commutes with t^μ and is contained in $G(\mathcal{O})$). From this formula, we deduce the following:

- The action of $\mathbf{G}_m$ preserves each semi-infinite orbit S_μ.
- If λ is dominant regular (that is, $\langle \alpha, \lambda \rangle > 0$ for each positive root α), then the action of $\mathbf{G}_m$ on S_μ extends to a monoid action

  $$a : \mathbf{A}^1 \times S_\mu \to S_\mu$$

  such that $a|_{\{0\} \times S_\mu}$ is the constant map taking the value $t^\mu G(\mathcal{O})$.

Proposition **??** can therefore be regarded as a special case of the following:

Proposition 24. *Let X be a variety (or inductive limit of algebraic varieties) equipped with an action of $\mathbf{G}_m$ which has a unique fixed point $x \in X$, and let $\mathscr{F}$ be a $\mathbf{G}_m$-equivariant constructible complex on X. If the $\mathbf{G}_m$-action of X extends to a monoid action*

$$a : \mathbf{A}^1 \times X \to X,$$

then the canonical maps

$$\mathscr{E}_x^! \to \mathrm{R}\Gamma_c(X, \mathscr{E}) \qquad \mathrm{R}\Gamma(X, \mathscr{E}) \to \mathscr{E}_x$$

are isomorphisms.

Proof. We will prove the second assertion; the first (which is the one we need for Proposition 23) then follows by Verdier duality. Let $\iota : \{x\} \to X$ be the inclusion map and let $\mathscr{K}$ be the fiber of the map $\mathscr{E} \to \iota_* \mathscr{E}_x$; we wish to show that the cohomology $\mathrm{H}^*(X, \mathscr{K})$ vanishes. Note that the pullback of $\mathscr{K}$ along the action map a is a complex of sheaves on $\mathbf{A}^1 \times X$ which is extended by zero from $\mathbf{G}_m \times X$ (since the stalk $\mathscr{K}_x$ vanishes), and its restriction to $\mathbf{G}_m \times X$ is isomorphic to $\underline{k}_{\mathbf{G}_m} \boxtimes \mathscr{K}$ (since $\mathscr{K}$ is $\mathbf{G}_m$-equivariant). It follows that pullback along a induces a map

$$\mathrm{H}^*(X, \mathscr{K}) \to \mathrm{H}^*(\mathbf{A}^1 \times X, \underline{k}_{\mathbf{G}_m} \boxtimes \mathscr{K}) \simeq \mathrm{H}^*(\mathbf{A}^1, \underline{k}_{\mathbf{G}_m}) \otimes \mathrm{H}^*(X, \mathscr{K}).$$

This map is automatically injective, since it has a left inverse given by evaluation at the point $1 \in \mathbf{A}^1$. The target vanishes (since the cohomology $\mathrm{H}^*(\mathbf{A}^1, \underline{k}_{\mathbf{G}_m})$ is trivial), so $\mathrm{H}^*(X, \mathscr{K})$ must vanish as well. $\square$

The proof of Proposition 24 offers another perspective on the semi-infinite orbits. Let X be an algebraic variety (or a directed limit of algebraic varieties) equipped with an action of $\mathbf{G}_m$. We define the *contracting locus* $X^+ \subseteq X$ to be characterized by the formula

$$\mathrm{Hom}(Y, X^+) := \mathrm{Hom}_{\mathbf{G}_m}(\mathbf{A}^1 \times Y, X) \subseteq \mathrm{Hom}_{\mathbf{G}_m}(\mathbf{G_m} \times Y, X) = \mathrm{Hom}(Y, X).$$

Stated more informally, if we write $a : \mathbf{G}_m \times X \to X$ for the action map, then $X^+ \subseteq X$ is the locus consisting of points $x \in X$ for which $a(s, x)$ has a limit as $s \in \mathbf{C}^*$ approaches zero. One can show that X^+ has roughly the same ontological status of X (for example, if X is an algebraic variety, then so is X^+). Moreover, the action of $\mathbf{G}_m$ on X restricts to an action of $\mathbf{G}_m$ on X^+, which extends to an action of the monoid $\mathbf{A}^1$ (this is immediate from the formula above).

Note that if X is a projective variety (or a directed limit of projective varieties), then every map $\mathbf{G}_m \to X$ extends to a map $\mathbf{A}^1 \to X$: that is, the inclusion $X^+ \hookrightarrow X$ is bijective on $\mathbf{C}$-valued points. Beware that it is not an isomorphism of schemes.

Example 25. Let $X = \mathbf{P}^1$ be the projective line, equipped with the “standard” action of $\mathbf{G}_m$ (which is simply transitive on $X \setminus \{0, \infty\}$. Then X^+ can be identified with the disjoint union $\mathbf{A}^1 \coprod \{\infty\}$.

Example 26. Let $X = \mathrm{Gr}_G$ be the affine Grassmannian, equipped with the $\mathbf{G}_m$-action determined by a regular dominant cocharacter $\lambda : \mathbf{G}_m \to T$. Then X^+ can be identified with the affine Grassmannian $\mathrm{Gr}_B \simeq \coprod_{\mu \in X_*(T)} S_\mu$.

Returning to the case of a general $\mathbf{G}_m$-action, we also have a *repelling locus* $X^- \subseteq X$, defined in a similar way

$$\mathrm{Hom}(Y, X^-) := \mathrm{Hom}_{\mathbf{G}_m}((\mathbf{P}^1 \setminus \{0\}) \times Y, X) \subseteq \mathrm{Hom}_{\mathbf{G}_m}(\mathbf{G_m} \times Y, X) = \mathrm{Hom}(Y, X).$$

Note that the intersection $X^\pm = X^+ \cap X^-$ parametrizes $\mathbf{G}_m$-equivariant maps $\mathbf{P}^1 \to X$. In particular, $X^\pm$ contains the fixed locus $X^{\mathbf{G}_m}$ as a summand (parametrizing constant maps from $\mathbf{P}^1$ to X). The inclusion $X^{\mathbf{G}_m} \hookrightarrow X^\pm$ is an isomorphism when X is affine (every map from $\mathbf{P}^1$ to an affine scheme is constant), but not general: if X is projective, then the inclusion $X^\pm$ is again bijective on points.

Example 27. Let $X = \mathrm{Spec}(R)$ be an affine scheme, so that the $\mathbf{G}_m$-action on X determines a grading $R = \bigoplus R_n$ of R. Then X^+ can be identified with the closed subscheme $\mathrm{Spec}(R/I_+)$, where $I_+ \subseteq R$ is the ideal generated by all homogeneous elements of negative degree. In other words, I_+ is the smallest graded ideal in R for which the induced grading on R/I_+ is concentrated in degrees ≥ 0 (so that the $\mathbf{G}_m$-action extends to $\mathbf{A}^1$). Similarly, $X^- = \mathrm{Spec}(R/I_-)$ where I_- is the ideal generated by homogeneous elements of positive degree, and $X^{\mathbf{G_m}} = X_- \cap X_+ = \mathrm{Spec}(R/(I_- + I_+))$.

Example 28. Let $X = \mathrm{Gr}_G$ be the affine Grassmannian, equipped with the $\mathbf{G}_m$-action determined by a regular dominant cocharacter $\lambda : \mathbf{G}_m \to T$. Then the repelling locus X^- can be identified with the affine Grassmannian Gr_{B^-}, where $B^- \supseteq T$ is the Borel subgroup opposite to T. Writing U^- for the unipotent radical of B^-, we have a decomposition

$$\mathrm{Gr}_{B^-} = \coprod_{\mu \in X_*(T)} T_\mu \qquad T_\mu := U^-(\mathcal{K}) \cdot t^\mu G(\mathcal{O}).$$

Remark 29. Since reductive algebraic groups are classified by their root data, the group scheme G always admits an automorphism σ which preserves the maximal torus T and acts by (-1) on $X_*(T)$; it follows that σ exchanges the Borel subgroups B and B^- (in some cases the automorphism σ is inner, given by conjugation by a representative of the "longest element" of the Weyl group of G). The automorphism σ induces an automorphism of Gr_G, which carries each of the semi-infinite orbits S_μ to the semi-infinite orbit $T_{-\mu}$ (and vice versa).

Let us now return to the proof of Theorem 17. Let $\mathscr{E}$ be an equivariant perverse sheaf on Gr_G. Proposition 20 then supplies an *upper bound* on the degrees in which the cohomology groups $\mathrm{H}^*_c(S_\mu, \mathscr{E}|_{S_\mu}$ are concentrated. We wish to prove the corresponding upper bound. Our starting point is the following:

Lemma 30. *Let $\mathscr{F}$ be an equivariant constructible complex on Gr_G which is perverse connective. Then, for every cocharacter $\mu \in X_*(T)$, the cohomology groups*

$$\mathrm{H}^*_c(T_\mu, \mathscr{F}|_{T_\mu})$$

are concentrated in degrees $* \leq -\langle 2\rho, \mu \rangle$.

Proof. By virtue of Remark 29, this is a restatement of Proposition 20 (applied to the semi-infinite orbit $S_{-\mu}$). □

Recall that a constructible complex $\mathscr{E}$ is perverse if and only if both $\mathscr{E}$ and its Verdier dual $\mathbf{D}\mathscr{E}$ are perverse connective. In this case, we can apply Lemma 30 to deduce that the cohomology groups

$$\mathrm{H}^*_c(T_\mu, (\mathbf{D}\mathscr{F})|_{T_\mu})$$

are concentrated in degrees $* \leq -\langle 2\rho, \mu \rangle$. Passing to vector space duals (and invoking Verdier duality), this yields the following:

Lemma 31. *Let $\mathscr{E}$ be an equivariant perverse sheaf on Gr_G (or, more generally, a constructible complex which is perverse coconnective). Let μ be a cocharacter of T. Then the cohomology groups*

$$\mathrm{H}^*_{T_\mu}(\mathscr{E}) := \mathrm{H}^*(T_\mu, \mathscr{E}|^!_{T_\mu})$$

are concentrated in degrees $* \geq \langle 2\rho, \mu \rangle$. Here $\mathscr{E}|^!_{T_\mu}$ denotes the complex $j^! \mathscr{E}$, where $j : T_\mu \hookrightarrow \mathrm{Gr}_G$ is the inclusion.

Note that for any point $x \in S_\mu \cap T_\mu$, its $\mathbf{G}_m$-orbit extends to a map $f : \mathbf{P}^1 \to \mathrm{Gr}_G$ carrying both $\{0\}$ and $\{\infty\}$ to the point x_μ. In this case, the map f must factor through S_μ and therefore be constant (since S_μ is a directed limit of affine spaces). It follows that x_μ is the unique intersection point of S_μ and T_μ. In particular, the intersection $S_\mu \cap T_\mu$ is proper, so that restriction to S_μ induces a comparison map

$$\theta : \mathrm{H}^*_{T_\mu}(\mathscr{E}) \to \mathrm{H}^*_c(S_\mu, \mathscr{E}|_{S_\mu}).$$

To complete the proof of Theorem 17, we need to show that the cohomology groups on the right hand side are concentrated in degrees $* \geq \langle 2\rho, \mu \rangle$. Lemma 31 asserts that analogous estimate hold for the left hand side. We are therefore reduced to showing that θ is an isomorphism. Thisis a special case of the following general principle:

Theorem 32 (Hyperbolic Localization (Braden, Goresky-MacPherson, Drinfeld-Gaitsgory))**.** *Let X be a variety equipped with an action of $\mathbf{G}_m$ having an isolated fixed point $x \in X$. Let X^+_x and X^-_x be the associated connected components of X^+ and X^- (the inverse image of x under maps $X^+ \to X^{\mathbf{G_m}}$ and $X^- \to X^{\mathbf{G}_m}$ given by evaluation at 0 and ∞, respectively). If $\mathscr{E}$ is a $\mathbf{G}_m$-equivariant constructible complex on X, then the restriction map*

$$\theta : \mathrm{H}^*_{X^-_x}(X, \mathscr{E}) \to \mathrm{H}^*_c(X^+_x, \mathscr{E}|_{X^+_x})$$

is an isomorphism.

In the situation of Theorem 32, the map θ factors as a composition

$$\mathrm{H}^*(X^-_x, \mathscr{E}|^!_{X^-_x}) \xrightarrow{\sim} \mathrm{H}^*((\mathscr{E}|^!_{X^-_x})_x) \xrightarrow{\widetilde{\theta}} \mathrm{H}^*((\mathscr{E}|_{X^+_x})^!_x) \xrightarrow{\sim} \mathrm{H}^*_c(X^+_x, \mathscr{E}|_{X^+_x})$$

where the outer isomorphisms are supplied by Proposition 24.

The construction of the map $\widetilde{\theta}$ is local on X with respect to the étale topology. Working locally, one can one can reduce to the case where $X = \mathrm{Spec}(R)$ is affine and $x \in X$ is the unique fixed point of $\mathbf{G}_m$ (by recent work of Alper-Hall-Rydh, this is true in great generality). In this case, $X^- = X^-_x$ and $X^+ = X^+_x$ are closed subschemes of X with intersection $X^- \cap X^+ = \{x\}$ (Example **??**).

Set $U^- = X \setminus X^-$, $U^+ = X \setminus X^+$, and $U^\pm = U^- \cap U^+$. If $\mathscr{E}$ is supported on X^- (or X^+), then Theorem 32 reduces to Proposition 24. Consequently, to prove Theorem 32, we can assume without loss of generality that $\mathscr{E}$ is the pushforward of its restriction to U^- (so that $\mathscr{E}|^!_{X^-} = 0$), and that $\mathscr{E}|_{U^-}$ is extended by zero from $U^\pm$ (so that $\mathscr{E}|_{X^+}$ is supported at the point x). In this case, the source of the morphism θ vanishes, and the target can be identified with (the cohomology of) the stalk $\mathscr{E}_x$. Theorem 32 is therefore equivalent to the assertion that the entire complex $\mathscr{E}$ is extended by zero from U^+. In other words, Theorem 32 can be summarized informally as saying that "extending by zero from U^+ commutes with pushing forward from U^-", at least for $\mathbf{G}_m$-equivariant sheaves.

Warning 33. In the statement of Theorem 32, the hypothesis that $\mathscr{E}$ is $\mathbf{G}_m$-equivariant can be weakened but not entirely omitted. For example, suppose that $X = \mathbf{A}^2$, equipped with the $\mathbf{G}_m$-action given by $(a, b) \mapsto (ta, t^{-1}b)$ (having a unique fixed point at the origin), and let $\mathscr{E}$ be the (nonequivariant!) sheaf on X obtained by pushing forward the constant sheaf $\underline{k}_{\mathbf{A}^1}$ along the diagonal $\mathbf{A}^1 \hookrightarrow X$. In this case, one can compute

$$\mathrm{H}^*_{X^-}(X, \mathscr{E}) = \begin{cases} k & \text{if } * = 2 \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{H}^*_c(X^+, \mathscr{E}|_{X^+}) = \begin{cases} k & \text{if } * = 0 \\ 0 & \text{otherwise.} \end{cases}$$