NOTES ON AFFINE GRASSMANNIANS

FELIKS RĄCZKA

1. Intro

You are reading the notes on affine grassmannians written for the GRT learning seminar at the IAS. The notes do not contain any original results and follow the exposition of Achar–Riche [1] and X. Zhu [2]. As a rule, we do not give complete proofs for most of the statements. Instead, we provide references, and we illustrate the main ideas of the proofs with concrete, easier to grasp examples.

2. Affine Grassmanian and Afffine Flag Variety

The goal of this section is to define (and establish the most basic properties of) the affine grassmannian and the affine flag variety.

2.1. **the Loop Group.** For a commutative ring R we denote by $R[[t]]$ (resp. $R((t)))$) the ring of formal (resp. Laurent) power series with coefficients in R.

Definition 2.1. Let G be a reductive group over $\mathbb{C}$.

(1) The functor $LG : \mathbb{C}\text{-alg} \to \text{Set}$ given by

$$LG(R) = G\left(R((t)))\right)$$

is called the *loop group* of G. It is represented by an ind-affine group ind-scheme over $\mathbb{C}$.

(2) The functor $L^{+}G : \mathbb{C}\text{-alg} \to \text{Set}$ given by

$$L^{+}G(R) = G\left(R[[t]]\right)$$

is called the *positive loop group* of G or the *arc group* of G. It is represented by an affine group scheme over $\mathbb{C}$.

(3) Let $\pi : L^{+}G \to G$ be the map induced by $t \mapsto 0$ and let $B \subset G$ be a Borel subgroup. Then the sub-group scheme

$$I = \pi^{-1}B \subset L^{+}G$$

is called the *Iwahori subgroup* associated with B.

Example 2.2. Let $G = GL_n$, and let $B \subset G$ be the Borel subgroup of lower-triangular matrices. $I(\mathbb{C})$ is the group of invertible matrices of form

$$\begin{pmatrix} a_{11} & ta_{12} & ta_{13} & \dots & ta_{1n} \\ a_{21} & a_{22} & ta_{23} & \dots & ta_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}, \quad a_{ij} \in \mathbb{C}[[t]],\ \det \in \mathbb{C}[[t]]^{\times}.$$

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2.2. **Affine Grassmannian and Affine Flag Variety.** We go straight to the central definition of these notes.

Definition 2.3. Let G be a reductive group over $\mathbb{C}$, let $B \subset G$ be Borel subgroup, and let $I \subset L^+G$ be the associated Iwahori subgroup.

(1) The *affine grassmannian* of G is the fppf sheafication of the functor $R \mapsto LG(R)/L^+G(R)$. We denote it by Gr_G.
(2) The *affine flag variety* of G is the fppf sheafication of the functor $R \mapsto LG(R)/I(R)$. We denote it by Flag_G.

We will not prove the following proposition (cf. Example 2.5(2) below).

Proposition 2.4 ([1, Section 2.2.1]). *The natural map* $\pi : \mathrm{Flag}_G \to \mathrm{Gr}_G$ *is a Zariski locally trivial fibration with fiber* G/B.

Example 2.5. We describe $\mathrm{Gr}_G(\mathbb{C})$ and $\mathrm{Flag}_G(\mathbb{C})$ for various G.

(1) If $G = \mathbb{G}_m$ then $\mathrm{Gr}_G(\mathbb{C}) = \mathrm{Flag}_G(\mathbb{C})$ is a discrete set indexed by the integers because
$$\mathbb{C}((t))^\times/\mathbb{C}[[t]]^\times \simeq \mathbb{Z}.$$
More generally, if $G = \mathbb{G}_m^{\times r}$ then $\mathrm{Gr}_G(\mathbb{C}) = \mathrm{Flag}_G(\mathbb{C})$ is a discrete set of points indexed by the co-characters of G.
(2) Let $G = \mathrm{GL}_n$. Set $V = \mathbb{C}((t))^{\oplus n}$. We say that a finitely generated $\mathbb{C}[[t]]$-submodule $\Lambda \subset V$ is a *lattice* if it is free and $\mathbb{C}((t)) \otimes_{\mathbb{C}[[t]]} \Lambda = V$. Then
$$\mathrm{Gr}_G(\mathbb{C}) = \{\Lambda \subset V : \Lambda \text{ is a lattice}\}.$$
This is because $LG(\mathbb{C})$ acts transitively on the set of lattices in V and $L^+G(\mathbb{C})$ is the stabilizer of the *standard lattice* $V_0 = \mathbb{C}[[t]]^{\oplus n}$. Similarly,
$$\mathrm{Flag}_G(\mathbb{C}) = \{\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_n \supset t\Lambda_1 : \Lambda_i \text{ is a lattice and } \dim_{\mathbb{C}} \Lambda_i/\Lambda_{i+1} = 1\},$$
because $I(\mathbb{C})$ is the stabilizer of the *standard flag*
$$\Lambda_0 = \mathbb{C}[[t]]^{\oplus n} \supset t\mathbb{C}[[t]] \oplus \mathbb{C}[[t]]^{\oplus n-1} \supset \cdots \supset t\mathbb{C}[[t]]^{\oplus n-1} \oplus \mathbb{C}[[t]] \supset t\mathbb{C}[[t]]^{\oplus n} = t\Lambda_0.$$
The map $\pi : \mathrm{Flag}_G \to \mathrm{Gr}_G$ is given by
$$x = [\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_n \supset t\Lambda_0] \mapsto \Lambda_1$$
and the fiber $\pi^{-1}(x)$ consists of full flags in $\Lambda_1/t\Lambda_1$.

Example 2.6. It is possible to promote the description of $\mathbb{C}$-points from Example 2.5 to an honest description of the functor Gr_G. Let us do that in the case $G = \mathrm{GL}_n$. For a $\mathbb{C}$-algebra R let $V_R = R((t))^{\oplus n}$. An *R-familly of lattices* in V_R is a finitely generated projective $R[[t]]$-submodule $\Lambda \subset V_R$ such that $R((t)) \otimes \Lambda = V_R$. Then
$$\mathrm{Gr}_G(R) = \{\Lambda \subset V_R : \Lambda \text{ is an } R\text{-familly of lattices in } V_R\}.$$
We will later interpret the right hand side as principal GL_n-bundles (i.e., vector bundles) on the formal disc $D_R = \mathrm{Spec}(R[[t]])$ with a trivialization on the punctured disc $D_R^* = \mathrm{Spec}(R((t)))$.

2.3. **Affine Grassmannian is an Ind-Projective Ind-Scheme.** The goal of this section is to sketch the proof of the following fundamental result.

Theorem 2.7. *Let G be a reductive group over $\mathbb{C}$. Then Gr_G is represented by an ind-projective ind-scheme.*

Sketch of the proof. The idea is to prove the theorem for $G = \mathrm{GL}_n$ first, and then reduce to this case by fixing a faithful representation $\rho : G \to \mathrm{GL}_n$ and arguing that the induced map $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$ is a closed embedding (see [2, Proposition 1.2.6] for the details). For $G = \mathrm{GL}_n$ we work with the description of GL_G from Example 2.6. For a non-negative integer $N \geq 0$ we define subfunctors

$$X_N \subset \mathrm{Gr}_G$$

by the formula

$$X_N(R) = \left\{\Lambda \in \mathrm{Gr}_G(R) : t^N \Lambda_{0,R} \subset \Lambda \subset t^{-N}\Lambda_{0,R}\right\},$$

where

$$\Lambda_{0,R} = R[[t]] \otimes_{\mathbb{C}} \Lambda_0 = R[[t]]^{\oplus n}$$

is the trivial R-familly of lattices. Then one shows that

(1) $\mathrm{Gr}_G = \bigcup_{N\geq 0} X_N$,
(2) X_N is represented by a projective scheme for every $N \geq 0$,
(3) The inclusion $X_N \subset X_{N+1}$ is represented by a closed embedding for every $N \geq 0$.

In Example 2.8 below we sketch the argument for (2). Formalizing it requires some extra work that we are not willing to do in these notes. □

Example 2.8. We explain how to naturally identify the sets

$$X_N(\mathbb{C}) = \left\{\Lambda : t^N \Lambda_0 \subset \Lambda \subset t^{-N}\Lambda_0\right\}$$

with some closed subsets in a disjoint union of 'usual' grassmannians. For a $\mathbb{C}$-vector space E let $\mathrm{Grass}(k, E)$ be the variety parametrizing k-dimensional subspaces of E. Consider the $\mathbb{C}$-vector space

$$V_N = t^{-N}\Lambda_0 / t^N \Lambda^0$$

and note that

$$\dim_{\mathbb{C}} V_N = 2Nn.$$

The multiplication by t on $t^{-N}\Lambda_0$ is $\mathbb{C}$-linear so it induces a $\mathbb{C}$-linear endomorphism of V_N that we denote θ_t. Now, define

$$\varphi_N : X_N(\mathbb{C}) \to \coprod_{j=0}^{2Nn} \mathrm{Grass}(j, V_N);\ \Lambda \mapsto \Lambda / t^N \Lambda_0.$$

Clearly, this map is injective. To show that the image is closed we proceed as follows. Fix a $\mathbb{C}$-vector space $\widetilde{W}$ satisfying

$$t^N \Lambda_0 \subset \widetilde{W} \subset t^{-N}\Lambda_0$$

and set $W = \widetilde{W}/t^N\Lambda_0$. Observe that $\widetilde{W}$ is a $\mathbb{C}[[t]]$-lattice in V if and only if $t\widetilde{W} \subset \widetilde{W}$ which is the case if and only if $\theta_t(W) \subset W$. Therefore

$$\varphi_N(X_N(\mathbb{C})) = \coprod_{j=0}^{2Nn} \left\{W \in \mathrm{Grass}(j, V_N) : \theta_t(W) \subset W\right\}.$$

Now, use Lemma 2.9 below.

Lemma 2.9. *Let E be a finite-dimensional $\mathbb{C}$-vector space and let* $\mathrm{Grass}(r, E)$ *be the grassmannian of r-dimensional subspaces of E. Let θ be an endomorphism of E and consider the set*

$$Z(\theta) = \{W \in \mathrm{Grass}(r, E) : \theta(W) \subset W\}.$$

Then $Z(\theta)$ is the set of zeroes of some global section $s_\theta \in H^0(\mathrm{Grass(r,E)}, T_{\mathrm{Grass(r,E)}})$. *In particular, $Z(\theta)$ is closed.*

Proof. Let $\mathcal{S}$ and $\mathcal{Q}$ be the tautological sub-bundle and the tautological quotient respectively. Then θ induces a global section s_θ of

$$T_{\mathrm{Grass(r,E)}} = \mathcal{H}om(\mathcal{S}, \mathcal{Q})$$

via

$$\mathcal{S} \hookrightarrow \mathscr{O}_{\mathrm{Grass(r,E)}} \otimes E \xrightarrow{\mathrm{Id}\otimes\theta} \mathscr{O}_{\mathrm{Grass(r,E)}} \otimes E \twoheadrightarrow \mathcal{Q}.$$

The fiber of $T_{\mathrm{Grass(r,E)}}$ at W is naturally identified with $\mathrm{Hom}_{\mathbb{C}}(W, E/W)$ and $s_{\theta|W}$ is given by

$$W \hookrightarrow E \xrightarrow{\theta} E \twoheadrightarrow E/W.$$

It follows that $\mathrm{Zeroes}(s_\theta) = Z(\theta)$. □

2.4. **Moduli description.** In this subsection we give an alternative description of Gr_G.

First, we introduce the auxiliary notation. For a $\mathbb{C}$-algebra R we define the (*punctured*) *formal disc* over R as $D_R = \mathrm{Spec}(R[[t]])$ (resp. $D^*_R = \mathrm{Spec}(R((t)))$). We also fix a curve $X/\mathbb{C}$ and a smooth closed point $x \in X$. We denote $X_R = X \times \mathrm{Spec}\ R$, $X^*_R = X^* \times \mathrm{Spec}\ R$.

Lemma 2.10. *Let G be a reductive group. The following functors* $\mathbb{C}\text{-alg} \to \mathrm{Sets}$ *are isomorphic.*

(1) *The functor*

$$R \mapsto \left\{(\mathscr{E}, \beta) : \mathscr{E} \text{ is a principal } G\text{-bundle}/D_R \text{ and } \beta \text{ is a trivialization of } \mathscr{E}_{|D^*_R}\right\},$$

(2) *The functor*

$$R \mapsto \left\{(\mathscr{E}, \beta) : \mathscr{E} \text{ is a principal } G\text{-bundle}/X_R \text{ and } \beta \text{ is a trivialization of } \mathscr{E}_{|X^*_R}\right\}.$$

About the proof. Since x is a smooth closed point, the (complement of x in the) formal completion of of X_R at x is D_R (resp. D^*_R). This defines a map from (2) to (1). The passage from (1) to (2) relies on the theorem of Beauville–Laszlo which asserts that we can glue a principal G-bundle $\mathscr{E}$ over D_R with the trivial G-bundle over X^*_R to a G-bundle over X_R using the trivialization β (see [2, Section 1.4] for details). □

We will not prove the following.

Proposition 2.11. *The functors from Lemma* 2.10 *are represented by* Gr_G.

Example 2.12. Let $G = \mathrm{GL}_n$. Then a principal G-bundle over Y is nothing else but a vector bundle over Y. In particular,

$$\{\text{Principal } G\text{-bundles } /D_R\} = \{\text{finitely generated, projective } R[[t]]\text{-modules}\}.$$

Let Λ be a finitely generated, projective $R[[t]]$-module. In the above description, the data of trivialization of the principal G-bundle is an isomorphism

$$\beta : R((t)) \otimes \Lambda \to R((t))^{\oplus n}.$$

In particular, β realizes Λ as a submodule of $R((t))^{\oplus n}$, so we can regard the pair (Λ, β) as an R-family of lattices. This recovers the description of Gr_G from Example 2.6.

3. Schubert varieties

The goal of this section is to define Schubert varieties inside affine grassmannians.

3.1. **Schubert varieties inside G/B.** The construction of Schubert varieties inside Gr_G is modeled on the classical construction of Schubert varieties inside the 'usual' flag variety G/B. To make the exposition in the affine case more transparent we recall briefly recall the classical construction.

Let G be a semi-simple, connected and simply connected algebraic group, let $B \subset G$ be a fixed Borel subgroup, and $T \subset B$ a fixed maximal torus. The Weyl group is $W = N(T)/T$. For an element $w \in W$ we denote by $\dot{w}$ its lift to G. Then

(1) The double coset $B\dot{w}B$ depends only on w and not on the lift $\dot{w}$,
(2) We have the *Bruhat decomposition*

$$G = \bigsqcup_{w\in W} B\dot{w}B.$$

Using the Bruhat decomposition we define for any $w \in W$ the *Schubert Cell*

$$C_w = B\dot{w}B/B \subset G/B,$$

and the *Schubert variety*

$$X_w = \overline{C_w} \subset G/B.$$

It is well-known that

$$X_w = \bigsqcup_{x\leq w} C_x,$$

where $\leq$ is the Bruhat-Chevalley order. For a parabolic subgroup $B \subset P$ it is possible to define Schubert cells and Schubert varieties inside the partial flag variety G/P in a similar way.

3.2. **Schubert varieties inside Gr_G.** For the rest of this section, we fix the standard datum of a reductive group G, Borel subgroup $B \subset G$, and a maximal torus $T \subset B$. We let

$$X_\bullet(T) = \mathrm{Hom}(\mathbb{G}_m, T)$$

be the lattice of co-characters partially ordered by

$$\alpha \leq \beta \iff \beta - \alpha \text{ is a sum of positive co-roots},$$

and we write $X_\bullet^+(T)$ for the set of dominant co-characters. Recall from one of the previous lectures (see also [2, 2.1.1]) that

$$L^+G(\mathbb{C})\backslash LG(\mathbb{C})/L^+G(\mathbb{C}) = X_\bullet^+(T). \tag{3.1}$$

Let $t \in \mathbb{C}[[t]]$ be the (obvious choice of the) uniformizer. For an element $\mu \in X_\bullet^+(T)$ we denote by t^μ the image of t under the map

$$\mathbb{G}_m(\mathbb{C}((t))) \xrightarrow{\mu} T(\mathbb{C}((t))) \hookrightarrow G(\mathbb{C}((t))) = LG(\mathbb{C}).$$

The bijection 3.1 may be realized explicitly as a Bruhat-like decomposition

$$LG(\mathbb{C}) = \bigsqcup_{\mu\in X_\bullet^+(T)} L^+G(\mathbb{C})t^\mu L^+G(\mathbb{C}).$$

By analogy to the classical case we define Schubert cells and Schbuert varieties inside the affine grassmannian.

Definition 3.1. With the above notation we define for $\mu \in X_\bullet^+(T)$

(1) The *Schubert cell*

$$\mathrm{Gr}_{G,\mu} = L^+Gt^\mu L^+G/L^+G \subset \mathrm{Gr}_G.$$

(2) The *Schubert variety*

$$\mathrm{Gr}_{G,\leq\mu} = \overline{\mathrm{Gr}_{G,\mu}} \subset \mathrm{Gr}_G$$

(the Zariski closure of the Schubert cell).

We will only sketch the proof of the following proposition.

Proposition 3.2. *With the above notation:*

(1) $\mathrm{Gr}_{G,\mu}$ *is a smooth, quasi-projective variety of dimension* $\langle 2\rho,\mu\rangle$*, where* ρ *is half-sum of the positive roots.*
(2) $\mathrm{Gr}_{G,\leq\mu} = \bigsqcup_{\lambda\leq\mu}\mathrm{Gr}_{G,\lambda}$.

Sketch of the proof. For the proof of (2) we refer to [1, Section 1.2.1.5]. We sketch the proof of (1). Let $[t^\mu]$ be the image of t^μ inside Gr_G. Then $\mathrm{Gr}_{G,\mu}$ is the L^+G-orbit of $[t^\mu]$. The stabilizer of $[t^\mu]$ is

$$L^+G \cap (t^\mu L^+Gt^{-\mu}),$$

and therefore

$$\mathrm{Gr}_{G,\mu} = L^+G/\left(L^+G \cap (t^\mu L^+Gt^{-\mu})\right).$$

Let $P_\mu \subset G$ be the parabolic subgroup corresponding to the roots satisfying $\langle\alpha,\mu\rangle \leq 0$. Then, the natural surjection $L^+G \to G$ induces a G-equivariant surjection

$$\pi : L^+G/\left(L^+G \cap (t^\mu L^+Gt^{-\mu})\right) \to G/P_\mu. \tag{3.2}$$

The fiber at eP_μ is easily seen to be an affine space and since G acts transitively on G/P_μ it follows that π is an affine fibration. This is enough to verify (1) (see examples below for further details). □

Example 3.3. We illustrate the content of Proposition 3.2(1) with explicit examples.

(1) Take $G = \mathrm{GL}_2$. Let $\alpha = e_1 - e_2$ be the positive root and let $\mu = m\alpha$ for some positive integer m. Then

$$(t^\mu L^+Gt^{-\mu})(\mathbb{C}) = \left\{\begin{pmatrix} a_{11} & t^{2m}a_{12} \\ t^{-2m}a_{21} & a_{22}\end{pmatrix} : a_{ij} \in \mathbb{C}[[t]]\right\},$$

and therefore

$$(L^+G \cap t^\mu L^+Gt^{-\mu})(\mathbb{C}) = \left\{\begin{pmatrix} a_{11} & t^{2m}a_{12} \\ a_{21} & a_{22}\end{pmatrix} : a_{ij} \in \mathbb{C}[[t]]\right\}.$$

We see that

$$P_\mu(\mathbb{C}) = \left\{\begin{pmatrix} * & 0 \\ * & *\end{pmatrix}\right\} \subset \mathrm{GL}_2(\mathbb{C})$$

Moreover, (3.2) maps a class $[g] \in \mathrm{Gr}_{G,\mu}(\mathbb{C})$ of $g \in G(\mathbb{C}[[t]])$ to its reduction mod t in $\mathbb{P}^1 = \mathrm{GL}_2/P_\mu$. The fiber of π at $\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ is

$$\left\{\begin{pmatrix} 1 & \sum_{j=1}^{2n-1} x_jt^j \\ 0 & 1\end{pmatrix} : x_1,\ldots,x_{2n-1} \in \mathbb{C}\right\} \simeq \mathbb{A}^{2n-1}(\mathbb{C}).$$

It follows that

$$\dim \mathrm{Gr}_{G,\mu} = 2n - 1 + 1 = 2n = \langle\mu,\rho\rangle.$$

(2) Let $G = \mathrm{GL}_n$ and let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n)$ be a dominant co-weight. Recall that the $\mathbb{C}$-points of Gr_G correspond to $\mathbb{C}[[t]]$-lattices inside $V = \mathbb{C}((t))^{\oplus n}$. The map

$$\pi : \mathrm{Gr}_{G,\mu} \to \mathrm{GL}_n / P_\mu$$

assigns to a lattice Λ a decreasing filtration on the $\mathbb{C}$-vector space $E = \Lambda_0 / t\Lambda_0$ via the formula

$$\pi(\Lambda) = \mathrm{Fil}^{\bullet}_{\Lambda} E; \quad \mathrm{Fil}^{i}_{\Lambda} E = t^{-\mu_i}\Lambda \cap \Lambda_0 / t^{-\mu_i}\Lambda \cap t\Lambda_0$$

(see [2, Example 2.1.12] for the details).

Remark 3.4 (cf. [2, Proposition 2.1.5]). Using the description of the stabilizer of $[t^\mu]$ it is easy to compute the tangent space of $\mathrm{Gr}_{G,\mu}$ at $t[^\mu]$. If $\mathfrak{g}$ is the Lie algebra of G then

$$T_{[t^\mu]}\mathrm{Gr}_{G,\mu} = \frac{\mathfrak{g}(\mathbb{C}[[t]])}{\mathfrak{g}(\mathbb{C}[[t]]) \cap \mathrm{Ad}_{[t^\mu]}\mathfrak{g}(\mathbb{C}[[t]])} = \bigoplus_{\langle \alpha, \mu \rangle} \frac{\mathfrak{g}_\alpha(\mathbb{C}[[t]])}{t^{\langle \alpha, \mu \rangle}\mathfrak{g}_\alpha(\mathbb{C}[[t]])}.$$

Remark 3.5 (Parity of the dimension of Schubert varieties). We know from Proposition 3.2 that $\dim \mathrm{Gr}_{G,\mu} = \dim \mathrm{Gr}_{G,\leq\mu} = \langle 2\rho, \mu \rangle$. If G is simply connected, then $\rho = \sum \omega_i$ is the sum of fundamental weights and therefore $\langle 2\rho, \mu \rangle \in 2\mathbb{Z}$. For arbitrary (reductive) G, the connected components of Gr_G are indexed by $X\bullet(T)/\mathbb{Z}\Phi^\vee$ (the algebraic fundamental group of G) and the parity of $\dim \mathrm{Gr}_{G,\mu}$ is constant on each connected component.

Now, we discuss the geometry of the Schubert varieties $\mathrm{Gr}_{G,\leq\mu}$ in some simple cases.

Definition 3.6. Let μ be a non-zero dominant co-weight.

(1) μ is called *minuscule* if $\langle \alpha, \mu \rangle \leq 1$ for all positive roots α.
(2) μ is called *quasi-minuscule* if it is not minuscule and $\langle \mu, \alpha \rangle \leq 2$ for all positive roots α.

Example 3.7. We describe $\mathrm{Gr}_{G,\leq\mu}$ for (quasi)-minuscule weights.

(1) If μ is minuscule then it is a minimal element of $X^+_\bullet(T)$ w.r.t. $\leq$. It follows that

$$\mathrm{Gr}_{G,\leq\mu} = \bigsqcup_{\lambda\leq\mu} \mathrm{Gr}_{G,\lambda} = \mathrm{Gr}_{G,\mu}.$$

On the other hand, for a minuscule weight we have

$$\mathrm{Gr}_{G,\mu} = G/P_u$$

(cf. the proof of Proposition 3.2).

(2) If μ is quasi-minuscule then it is a minimal element of $X^+_\bullet(T) \smallsetminus \{0\}$ and we have

$$\mathrm{Gr}_{G,\leq\mu} = \bigsqcup_{\lambda\leq\mu} \mathrm{Gr}_{G,\lambda} = \mathrm{Gr}_{G,0} \sqcup \mathrm{Gr}_{G,\mu}.$$

Assume that G is semi-simple. Then μ is a short dominant co-root and $\mathrm{Gr}_{G,\mu}$ can be identified with the total space of a line bundle $\mathscr{L} \to G/P_\mu$ parabolically induced from the dominant root $\mu^\vee$ (see [2, Lemma 2.1.14] for details).

3.3. **Moduli description.** We end this section with a description of the Schubert varieties in terms of the moduli interpretation of Gr_G.

Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two principal G-bundles over $D = \mathrm{Spec}(\mathbb{C}[[t]])$ and let

$$\beta : \mathscr{E}_{1|D^*} \to \mathscr{E}_{2|D^*}$$

be an isomorphism over $D^* = \mathrm{Spec}(\mathbb{C}((t)))$. Let $\mathscr{E}^0$ be the trivial principal G-bundle, and let

$$\phi_i : \mathscr{E}_{i|D^*} \to \mathscr{E}^0_{|D^*}$$

be trivializations over D^*. Then

$$\phi_2 \beta \phi_1^{-1} \in \mathrm{Aut}(\mathscr{E}^0_{D^*}) = LG(\mathbb{C})$$

determines an element

$$\mathrm{Inv}(\beta) = L^+G(\mathbb{C}) \backslash LG(\mathbb{C}) / L^+G(\mathbb{C}) = X_\bullet^+(T)$$

that depends only on β (and not on ϕ_i). Recall that we can interpret elements of $\mathrm{Gr}_G(\mathbb{C})$ as pairs $(\mathscr{E}, \beta)$, where $\mathscr{E}$ is a principal G-bundle over D and $\beta : \mathscr{E}_{|D^*} \to \mathscr{E}^0_{D^*}$ is an isomorphism.

Proposition 3.8. *We have*

$$\mathrm{Gr}_{G,\leq\mu}(\mathbb{C}) = \{(\mathscr{E}, \beta) : \mathrm{Inv}(\beta) \leq \mu\}.$$

4. Semi-Infinite Orbits

In this final section, we briefly discuss semi-infinite orbits.

First, recall the *Iwasawa decomposition*

$$LG(\mathbb{C}) = LB(\mathbb{C}) L^+G(\mathbb{C}), \tag{4.1}$$

where $B \subset G$ is a Borel subgroup. If we write $B = N \rtimes T$, where $T \subset B$ is a maximal torus and N is the unipotent radical then LN acts on the left on Gr_G and (4.1) gives rise to a decomposition

$$\mathrm{Gr}_G = \bigsqcup_{\mu X_\bullet(T)} LN.[t^\mu].$$

We denote

$$S_\mu = LN.[t^\mu], \qquad S_{\leq\mu} = \overline{S_\mu}.$$

We remark that contrary to the case of Schubert varieties, S_μ will not in general have finite dimensions.

Proposition 4.1 ([2, Proposition 5.3.6, Theorem 5.3.9])**.** *With the above notation:*

1. $S_{\leq\mu} = \bigcup_{\lambda\leq\mu} S_\lambda$
2. $\mathrm{Gr}_{G,\leq\lambda} \cap S_\mu \neq \varnothing$ *if and only if* $[t^\mu] \in \mathrm{Gr}_{G,\leq\lambda}$. *If this is the case, then the intersection has pure dimension* $\langle \rho, \lambda + \mu \rangle$.

To end this note, we present an alternative way to define S_μ. First, observe that the construction of Gr_H makes sense (and is functorial) for any, not necessarily reductive algebraic group H (note however that we need reductive, for example, to prove that Gr_H is ind-projective). From the semi-direct decomposition $B = N \rtimes T$ we obtain two homomorphisms

$$T \xleftarrow{q} B \xrightarrow{\iota} G$$

that induce morphisms of the corresponding affine grassmannians

$$\mathrm{Gr}_T \xleftarrow{q} \mathrm{Gr}_B \xrightarrow{\iota} \mathrm{Gr}_G.$$

Recall from Example 2.5(1) that $\mathrm{Gr}_T(\mathbb{C})$ is a discrete set indexed by $X_\bullet(T)$

$$\mathrm{Gr}_T(\mathbb{C}) = \{L_\mu : \mu \in X_\bullet(X)\}.$$

It is known (see [1, Section 1.2.3], [2, Formula 5.3.5]) that

(1) $q : \mathrm{Gr}_B \to \mathrm{Gr}_T$ is a bijection on the level of connected components,
(2) $\iota : \mathrm{Gr}_B \to \mathrm{Gr}_G$ is a bijection on points (but is very far from being an isomorphism).

The alternative definition of S_μ is given in the proposition below.

Proposition 4.2. $S_\mu = \iota(q^{-1}(L_\mu))$.

Note that because of (1)-(2) above, we recover the decomposition

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in X_\bullet(T)} S_\mu.$$

References

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Institute for Advanced Study, 1 Einstein dr, Princeton, NJ
Email address: `fraczka@impan.pl`