FUSION PRODUCTS AND THE COMMUTATIVITY CONSTRAINT

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We explain how convolution Beilinson-Drinfeld Grassmanians allows to prove the commutativity constraint on $\mathrm{Sat} = \mathrm{Perv}(\mathcal{H})$, following [Zhu] quite closely.

1. Beauville-Laszlo

Let k be a ring. Recall that we fixed a reductive group $G/k[\![t]\!]$ and defined the following ind-schemes over k:

$$\begin{aligned} L^+G(R) &= G(R[\![t]\!]) \\ LG(R) &= G(R(\!(t)\!)) \\ \mathrm{Gr}_G &= LG/L^+G \end{aligned}$$

where the last quotient is made as fpqc (or equivalently, since everything is smooth, étale) sheaves. There is an easy moduli description of Gr_G. If R is a ring, we let $D_R = \operatorname{Spec} R[\![t]\!]$ denote the formal disk and $D_R^\times = \operatorname{Spec} R(\!(t)\!)$ the formal punctured disk.

Proposition 1. *Gr_G represents the functor*

$$R \mapsto \{(\mathcal{E}, \nu) \mid \mathcal{E} \text{ is a } G\text{-bundle on } D_R, \nu : \mathcal{E}|_{D_R^\times} \simeq G \times D_R^\times \text{ is a trivialization}\}/\text{isom.}$$

To prove this, we need

Lemma 2. *A G-bundle on D_R splits over $D_{R'}$ for some étale cover R' of R.*

Proof. Let $\mathcal{E}$ be a G-bundle on D_R. Since G is smooth, the same goes for $\mathcal{E}$ (they are étale-locally the same) so $\mathcal{E} \otimes_{D_R} \operatorname{Spec} R \to \operatorname{Spec} R$ admits a section over some finite étale cover R' of R (smooth morphisms look étale-locally like affine space, which admit sections). Now, since $\mathcal{E}$ is smooth, by the infinitesimal criterion, we can lift this section

$$\operatorname{Spec} R' \to \mathcal{E} \otimes_{D_R} \operatorname{Spec} R'$$

to a section

$$\operatorname{Spec} R'[t]/(t^2) \to \mathcal{E} \otimes_{D_R} \operatorname{Spec} R'[t]/(t^2),$$

which we then lift to a section

$$\operatorname{Spec} R'[t]/(t^3) \to \mathcal{E} \otimes_{D_R} \operatorname{Spec} R'[t]/(t^3),$$

etc. Taking the limit (note that everything is affine), we obtain the wanted section $\mathcal{E} \times_{D_R} D_{R'} \to D_{R'}$. $\square$

Proof of Proposition 1. Call this functor F. Note that the loop group LG represents the functor

$$R \mapsto \{(\mathcal{E}, \nu, \beta) \mid \mathcal{E} \text{ is a } G\text{-bundle on } D_R, \nu : \mathcal{E}|_{D_R^\times} \simeq G \times D_R^\times,\ \beta : \mathcal{E} \simeq G \times D_R\}/\text{isom}$$

so we get a map from LG to F. This might be a bit confusing, but the point is that all trivialised G-bundles $(\mathcal{E}, \beta)$ are isomorphic, so only ν matters:

$$(\mathcal{E}, \nu, \beta) \mapsto \beta|_{D_R^\times} \circ \nu^{-1}$$

is the reverse map. Moreover, the action of L^+G on LG translates via this interpretation to acting on β. We need to show that this map $LG \to F$ is an L^+G-bundle (so that $F \simeq LG/L^+G$). To see that $1 \to L^+G \to LG \to F \to 1$ is exact, it suffices to check this on stalks for the étale topology. In other words, we may assume that R is strictly henselian. Then, by Lemma 2 all bundles are trivial and it is clear that $F(R) = LG(R)/LG^+(R)$ (since L^+G acts transitively on the set of β, for a fixed $(\mathcal{E}, \nu)$). $\square$

It turns out that there is also a more global interpretation of Gr_G.

Theorem 3. *Let X/k be a smooth curve and $x_0 \in X$ a point with local coordinate t, so that the formal ring of X at x is isomorphic to $k[\![t]\!]$. Then, Gr_G represents the functor*

$$R \mapsto \{(\mathcal{E}, \nu) \mid \mathcal{E} \textit{ is a } G\textit{-bundle on } X_R, \nu : \mathcal{E}|_{(X - \{x_0\})_R} \simeq G \times (X - \{x_0\})_R)\}/\textit{isom.}$$

Sketch. We wish to "glue" the trivial bundle over $X - \{x_0\}$ with a bundle over D_{x_0} (the formal disk centered at x) to get a bundle over X. This problem is Zariski local and in fact, by spreading out, we reduce to the case $X = \operatorname{Spec} \mathcal{O}_{X,x_0}$ local. Since G is affine, G-bundles correspond to quasicoherent algebras so this reduces to a result in commutative algebra: Beauville and Lazslo show that if A is a local ring and f a non-zero divisor, modules over A_f and $\widehat{A}$ that agree over $\widehat{A}_f$ glue to modules over A. Here $\widehat{A}$ is the f-adic completion of A. $\square$

2. Convolution and twisted box products

Recall also that we defined the Satake category Sat as $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G) = \mathrm{Perv}(\mathcal{H})$ where $\mathcal{H}$ is the Hecke stack $[L^+G \backslash LG / L^+G]$. (There were some technicalities involved because Gr_G is infinite-dimensional, but this is what it amounted to in the end.) Convolution was defined by interpreting $\mathcal{H}$ as $BL^+G \times_{BLG} BL^+G$ but we wish to reinterpret this definition in the world of ind-schemes.

Before we do this, let us start with an observation/definition. If $E \xrightarrow{p} X$ is an H-bundle and Y is an H-space, we define the **twisted product** $X\tilde{\times}Y$ as the quotient $E\times^H Y = (E\times Y)/H$ where H acts diagonally (if both actions are right-actions; if one is a right-action and the other a left-action, we act via $g(e,y) = (eg, g^{-1}y)$). Moreover, since $[E/H] = X$, p induces an isomorphism

$$D^b_{c,H}(X) \simeq D^b_{c,H}(E).$$

which we can shift to get an isomorphism

$$p^*[\dim H] : \mathrm{Perv}(X) \to \mathrm{Perv}_H(E).$$

Now, if $\mathcal{F}$ is a sheaf on E and $\mathbb{G}$ is an H-equivariant sheaf on Y, we define their **twisted box product** $p^*\mathcal{F}\tilde{\boxtimes}\mathcal{G}$ as follows: the box product $p^*\mathcal{F}\boxtimes\mathcal{G}$ is an H-equivariant sheaf on $E\times Y$ so descends to $X\tilde{\times}Y$.

Consider now the case $H = L^+G$, $E = LG$, $X = \mathrm{Gr}_G$ and $Y = \mathrm{Gr}_G$. The twisted product $\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G$ is called the **convolution Grassmanian**. We have a map

$$m : \mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G = LG\times^{L^+G}\mathrm{Gr}_G \to \mathrm{Gr}_G$$

induced by multiplication: $(g,[h]) = [gh]$. We define convolution as

$$\mathcal{F}_1\star\mathcal{F}_2 := m_!(\mathcal{F}_1\tilde{\boxtimes}\mathcal{F}_2).$$

For later use, we note

Lemma 4. *The functor of points of* $\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G$ *is*

$$R\mapsto\{(\mathcal{E},\mathcal{F},\nu,\beta)\mid \mathcal{E},\mathcal{F}\ G\text{-bundles on } D_R, \nu:\mathcal{E}|_{D_R^\times}\simeq G\times D_R^\times, \beta:\mathcal{E}|_{D_R^\times}\simeq\mathcal{F}|_{D_R^\times}\}.$$

Proof. Call this functor $\tilde{F}$. Recall that $LG\times\mathrm{Gr}_G\to\tilde{F}$ represents pairs of bundles on the disk $(\mathcal{E},\mathcal{F})$ together with trivialisations ν and μ of $\mathcal{E}$ and $\mathcal{F}$ on $D_R^\times$, and a trivialisation β of $\mathcal{E}$ on D_R. The action of L^+G is by acting on both β and μ. In particular, the isomorphism $\mu^{-1}\circ\beta|_{D_R^\times}:\mathcal{E}^\times_{D_R}\to\mathcal{F}^\times_{D_R}$ is fixed by this action. We obtain a map

$$LG\times\mathrm{Gr}_G\to\tilde{F},\ (\mathcal{E},\mathcal{F},\nu,\mu,\beta)\mapsto(\mathcal{E},\mathcal{F},\nu,\mu^{-1}\circ\beta|_{D_R^\times}:\mathcal{E}^\times_{D_R}\to\mathcal{F}^\times_{D_R})$$

which realises $LG\times\mathrm{Gr}_G$ as a L^+G-bundle over $\tilde{F}$ (we check it as before using Lemma 2.) □

We note that $\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G$ is also obviously isomorphic to $\mathrm{Gr}_G\times\mathrm{Gr}_G$ as choosing β amounts to choosing a trivialisation $\mu=\nu\beta^{-1}:\mathcal{F}|_{D_R^\times}\simeq G\times D_R^\times$. In terms of elements, this corresponds to the map $LG\times^{L^+G}\mathrm{Gr}_G\to\mathrm{Gr}_G\times\mathrm{Gr}_G$ given by $([g],[h])\mapsto([g],[gh])$. With this isomorphism, m becomes the second projection, i.e.

$$m(\mathcal{E},\mathcal{F},\nu,\beta)=(\mathcal{F},\nu\beta^{-1}).$$

In all of the above, we may also replace D_R and $D_R^\times$ by X and $X - \{x_0\}$ as before using formal (Beauville-Laszlo) glueing.

Remark 5. Let us explain why our definition of convolution recovers the one of the previous talk. In our definition, we pullback L^+G-equivariant sheaves on Gr_G to $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$, i.e. we pullback sheaves on $\mathcal{H} = [L^+G \backslash \mathrm{Gr}_G]$ to

$$[L^+G \backslash \mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G].$$

Here, the action of L^+G is on the first factor only of $LG \times^{L^+G} \mathrm{Gr}_G$ (or the last factor only, this is the same). To see this, since $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \simeq \mathrm{Gr}_G \times \mathrm{Gr}_G$, we more generally prove that

$$[L^+G \backslash \mathrm{Gr}_G \tilde{\times} \dots \tilde{\times} Gr_G] \simeq BL^+G \times_{BLG} \dots \times_{BLG} BL^+G$$

where $\mathrm{Gr}_G^{\tilde{\times} n}$ is the convolution Grassmanian as defined in [Zhu, (1.2.15)], representing tuples

$$(\mathcal{E}_1, \dots, cE_n, \nu, \alpha_1, \dots, \alpha_{n-1})$$

where $\mathcal{E}_1, \dots, \mathcal{E}_n$ are G-bundles on D, $\nu : \mathcal{E}_1|_{D^\times} \simeq G \times D^\times$ is a trivialisation and $\alpha_i : \mathcal{E}_i|_{D^\times} \simeq \mathcal{E}_{i+1}|_{D^\times}$. Here, L^+G acts on ν. Indeed, we may add to this moduli problem an extra trivialised bundle $\mu : \mathcal{E}_0 \simeq G \times D$ (this is unique up to isomorphism so doesn't change anything). Then, instead of the data ν, we ask for an isomorphism $\alpha_0 = \nu^{-1}\mu|_{D^\times} : \mathcal{E}_0 \simeq \mathcal{E}_1|_{D^\times}$. The left-action of L^+G on ν is the same as a right-action of L^+G on μ (trivial on the α_i), and we pass to the quotient we obtain $n+1$-tuples $(\mathcal{E}_0, \dots, \mathcal{E}_n)$ of bundles together with isomorphisms on $D^\times$, just like claimed. (Note that for $n = 1$ this recovers the isomorphism $\mathcal{H} \simeq BL^+G \times_{BLG} BL^+G$ of the previous talk.)

3. Beilinson-Drinfeld Grassmanians and fusion

As before, we fix a smooth curve X/k, but this time we do not fix a basepoint. The Beilinson-Drinfeld Grassmanian is the following functor:

$$\mathrm{Gr}_{G,X} : R \mapsto \{(\mathcal{E}, \nu, x \in X(R)) \mid \mathcal{E} \text{ is a } G\text{-bundle on } X_R, \nu : \mathcal{E}|_{X_R - \{x\}} \simeq G \times (X_R - \{x\})\}.$$

Here, by $X_R - \{x\}$, we mean the open subset $X_R - \Gamma_x$ where $\Gamma_x \subseteq X_R$ is the graph of x (equivalently, the subscheme defined by the closed immersion $x : \operatorname{Spec} R \to X_R$). This is an ind-projective scheme over X.

In this setting, an extension of Beauville-Laszlo glueing allows us to identify $\mathrm{Gr}_{G,X}$ with the quotient L_XG/L_X^+G where

$$L_X^+G(R) = \{(x \in X(R), g \in G(D_x))\}, \; L_XG(D_x^\times) = \{(x \in X(R), g \in G(\widehat{\Gamma}_x - \Gamma_x))\}$$

are loop and arc groups over X. Here, the formal disk $D_x \simeq D_R$ is defined as the infinitesimal neighbourhood of x in X_R (the formal completion of the graph Γ_x along itself) viewed as an affine scheme instead of a formal scheme. Then, $D_x^\times = D_x - \{x\}$.

We similarly define

$$\mathrm{Gr}_{G,X^n} = \{(\mathcal{E}, \nu, x \in X^n(R)) \mid \nu_i : \mathcal{E}|_{X_R - \{x\}} \simeq G \times X_R - \{x\}\}$$

where $\{x\} = \{x_1, \dots, x_n\}$ for $x = (x_1, \dots, x_n)$. This is a similar quotient of loop groups over X^n (where we take graphs in X^n).

Since at the end of the day we are interested only in local properties even if we need to make a detour through global geometry, we might as well fix $X = \mathbb{A}^1_k$. Then, since we have a global coordinate t which gives local coordinates at every point, we get

$$\mathrm{Gr}_{G,X} \simeq \mathrm{Gr}_G \times X,\ L_X G \simeq LG \times X,\ L^+_X G \simeq L^+G \times X.$$

In general $\mathrm{Gr}_{G,X}$ is some twisted product of Gr_G with X, see [Zhu, (3.1.10)].

However, Gr_{G,X^2} is still quite different from $\mathrm{Gr}_G \times X^2$, and this will be key for us. More precisely, we have the following.

Lemma 6. *Let $\Delta : X \to X^2$ be the diagonal. Then,*

$$\mathrm{Gr}_{G,X^2} \times_{X^2,\Delta} X \simeq \mathrm{Gr}_{G,X}$$

and

$$\mathrm{Gr}_{G,X^2} \times_{X^2} (X^2 - \Delta(X)) \simeq \mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}.$$

Proof. The restriction to the diagonal is clear. For the second part, observe that giving a bundle $\mathcal{E}$ trivialised outside $\{x_1, x_2\}$ is the same as giving one bundle trivialised $\mathcal{E}_1$ outside $\{x_1\}$ (obtained by glueing $\mathcal{E}|_{X-\{x_1\}}$ with the trivial bundle on $X - \{x_2\}$) and one trivialised outside $\{x_2\}$ (in the same way): they then glue to give back $\mathcal{E}$. □

Note that the dimension of fibers over X^2 goes up under generisation, contrary to what happens for varieties! This is because we are in an infinite-dimensional setting and these are only ind-schemes.

We will define convolution on the Beilinson-Drinfeld Grassmanian analogously to what we did in the previous section. We start with the $L^+_X G$-bundle over $\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X} = L_X G \times \mathrm{Gr}_{G,X}$. As before, this has the moduli interpretation (thanks to Beauville-Laszlo which lets us replace trivialisations over D_{x_i} with trivialisations over $X - \{x_i\}$)

$$R \mapsto \{ \begin{array}{c} (\mathcal{F}_1, \nu_1, \mu_1, \mathcal{F}_2, \nu_2, x_1 \in X(R), x_2 \in X(R)) \mid \\ \nu_i : \mathcal{F}_i|_{X_R - \{x_i\}} \simeq G \times (X_R - \{x_i\}), \mu_1 : \mathcal{F}_1|_{X_R - \{x_2\}} \simeq G \times (X_R - \{x_2\})\} \end{array}.$$

This gives us a convolution Grassmanian

$$\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X} = L_X G \times^{L^+_X G} \mathrm{Gr}_{G,X}$$

over X^2 classifying

$$\{(\mathcal{E}, \mathcal{F}, \nu, \alpha) \mid \nu : \mathcal{E}|_{X_R - \{x_1\}} \simeq G \times (X_R - \{x_1\}), \alpha : \mathcal{E}_{X_R - \{x_1\}} \simeq \mathcal{F}|_{X_R - \{x_2\}}\}$$

together with a convolution map from the convolution Grassmanian

$$m : \mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X} \to \mathrm{Gr}_{G,X^2}$$

defined by

$$m(\mathcal{E}, \mathcal{F}, \nu, \alpha, x_1, x_2) = (\mathcal{F}, \beta \circ \nu^{-1}, x_1, x_2).$$

Finally, we define convolution of sheaves on the Beilinson-Drinfeld Grassmanian

$$* : D^b_{c,L^+_X G}(\mathrm{Gr}_{G,X})^2 \to D^b_{c,L^+_{X^2} G}(\mathrm{Gr}_{G,X^2})$$

by

$$\mathcal{F} * \mathcal{G} := m_!(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2).$$

When restricted to the diagonal, all of this coincides with our previous construction. In particular, if p denotes the projection $\mathrm{Gr}_{G,X} = \mathrm{Gr}_G \times X \to \mathrm{Gr}_G$ and $i : \mathrm{Gr}_{G,X} \to \mathrm{Gr}_{G,X^2}$ is the base change of the diagonal map $\Delta : X \to X^2$, we get the following compatibility between our convolutions

Lemma 7. *For any $\mathcal{F}, \mathcal{G} \in D^c_{b,L^+G}(\mathrm{Gr}_G)$, we have*

$$i^*(p^*\mathcal{F} * p^*\mathcal{G}) = p^*(\mathcal{F} \star \mathcal{G}) \in D^b_c(\mathrm{Gr}_G \times X).$$

Proof. Leet m' denote the multiplication map $\mathrm{Gr}_G \tilde{\mathrm{Gr}}_G \to \mathrm{Gr}_G$ and i' the diagonal map $\mathrm{Gr}_{G,X} \to \mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}$. The proof is routine and follows from usual base change theorems. Proper base change allows us to commute $m_!$ with i^*:

$$i^*(p^*\mathcal{F} * p^*\mathcal{G}) = i^* m_!(p^*\mathcal{F} \tilde{\boxtimes} p^*\mathcal{G}) = (m' \times \mathrm{id}_X \times \mathrm{id}_X)_! i'^*(p^*\mathcal{F} \tilde{\boxtimes} p^*\mathcal{G}).$$

Let us first consider the nontwisted box product as a usual box product

$$p^*\mathcal{F} \boxtimes p^*\mathcal{G} = p_1^* p^*\mathcal{F} \otimes p_2^* p^*\mathcal{G}$$

where p_1, p_2 are the two projections $L_X G \times \mathrm{Gr}_{G,X} \to L_X G, \mathrm{Gr}_{G,X}$. Since $p \circ p_1 \circ i' = p'_1 \circ (p \times p)$ and $p \circ p_2 \circ i' = p'_2 \circ (p \times p)$ where p'_1, p'_2 are the analogous projections $LG \times \mathrm{Gr}_G \to LG, \mathrm{Gr}_G$, we get

$$i'^*(p^*\mathcal{F} \boxtimes p^*\mathcal{G}) = (p \times p)^*(\mathcal{F} \boxtimes \mathcal{G}).$$

Since pullbacks commute with quotients, we obtain

$$i'^*(p^*\mathcal{F} \tilde{\boxtimes} p^*\mathcal{G}) = (p \times p)^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}).$$

Finally, we have

$$(m' \times \mathrm{id}_X \times \mathrm{id}_X)_!(p')^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) = p^* m'_!(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) = \mathcal{F} \star \mathcal{G}$$

again by proper base change. $\square$

4. UNIVERSALLY LOCALLY ACYCLIC MORPHISMS

In general, convolution does not map $\mathrm{Perv}_{L^+_X G}(\mathrm{Gr}_{G,X})$ to $\mathrm{Perv}_{L^+_{X^2} G}(\mathrm{Gr}_{G,X^2})$ (e.g. if G is trivial, these are just categories of perverse sheaves on X and convolution is the box product). For this reason, we will introduce subcategories Sat_{X^n}, which will then turn out to be preserved under convolution.

If S is a scheme and s is a geometric point, we let $S_{(s)}$ denote its strict Henselisation at s.

Definition 8. *Let morphism $f : X \to S$ be a morphism. A sheaf $\mathcal{F} \in D^b_c(X)$ is **locally acyclic** (relatively to f) at a geometric point x if for every geometric point $t \in S_{(f(x))}$, the natural map*

$$R\Gamma(X_{(x)}, \mathcal{F}) \to R\Gamma((X_{(x)})_t, \mathcal{F})$$

is an isomorphism, where $(X_{(x)})_t$ is the fiber of $X_{(x)} \to S_{(f(x))}$ at t. It is locally acyclic if it is everywhere locally acyclic, and universally locally acyclic (ULA) if it is locally acyclic after every base change.

Remark 9. Gabber proved that locally acyclic morphisms are universally locally acyclic if S is Noetherian.

The ULA condition (combined with more usual perversity conditions, e.g. perversity on geometric fibers) may be thought of as some kind of "relative perversity condition", see [HS]. In this note, wee will content ourselves with the following categories of "relatively perverse" sheaves.

Definition 10. *We let Sat_{X^n} denote the subcategory of $\mathrm{Perv}_{L^+_{X^n} G}(\mathrm{Gr}_{G,X^n})$ consisting of sheaves that are ULA with respect to the projection $\mathrm{Gr}_{G,X^n} \to X^n$.*

Local acyclicity may be stated in terms of nearby/vanishing cycles. Let us write $t \to s$ if $s \in S$ is a geometric point and $t \in S_{(s)}$ is a geometric point.

Lemma 11. *$f : X \to S$ is locally acyclic if and only if*

$$\mathcal{F}|_{X_s} \overset{\sim}{\to} \Psi_{t\to s}(\mathcal{F})$$

for every geometric specialisation $t \to s$. Here,

$$\Psi_{t\to s}(\mathcal{F}) = j_{t*} j_t^* \mathcal{F}|_{X_s} = i_s^* j_{t*} j_t^* \mathcal{F}$$

is the sheaf of nearby cycles of $j^{t*}\mathcal{F}$along the specialisation $t \to s$, where $j_t : X \times_S t \to X$ and $i_s : X \times s \to X$.

Proof. We need to check that all the geometric fibers are the same. Let x be a geometric point of X_s. Then,

$$\Psi_{t\to s}(\mathcal{F})_x = (j_{t*}j_t^*\mathcal{F})_x = \lim_{U\ni x} R\Gamma(U, j_t^*\mathcal{F}) = \lim_{U\ni x} R\Gamma(U\times_S t, \mathcal{F}).$$

Since étale cohomology commutes with limits (with affine q.c. transitions), this is the same as

$$R\Gamma(X_{(x)}\times t, \mathcal{F}).$$

□

We summarise the properties of ULA morphisms we will use. Note that any morphism to a point is ULA since there is nothing to check in this case.

Lemma 12. *(i) $\mathcal{F}$ is ULA with respect to $\mathrm{id} : S\to S$ if and only if its cohomology sheaves are local systems.*

(ii) If $f : X\to Y$ is a proper map over S and $\mathcal{F}$ is ULA with respect to $X\to S$, then $f_*\mathcal{F}$ is ULA with respect to $Y\to S$.

(iii) If $f : X\to Y$ is a proper map over S, then $f^*\mathcal{F}$ is ULA with respect to $X\to S$ if and only if $\mathcal{F}$ is ULA with respect to $Y\to S$.

(iv) If $f_i : X_i\to S$, $i = 1, 2$ are two morphisms and $\mathcal{F}_i$ is ULA with respect to $X_i\to S$, then $\mathcal{F}_1\boxtimes\mathcal{F}_2$ is ULA with respect to $f_1\times_S f_2 : X_1\times_S X_2\to S$.

(v) If $f_i : X_i\to S_i$, $i = 1, 2$ are two morphisms and $\mathcal{F}_i$ is ULA with respect to $X_i\to S$, then $\mathcal{F}_1\boxtimes\mathcal{F}_2$ is ULA with respect to $f_1\times_S f_2 : X_1\times_S X_2\to S$.

Proof. If $\mathcal{F}$ is an actual sheaf (instead of a sheaf complex), (i) follows from [Stacks, Tag 0GKC] (a sheaf with stalks constant along specialisations is locally constant). The general case follows by induction on the cohomological amplitude of $\mathcal{F}$ (we can write $\mathcal{F}$ as an extension of a shifted sheaf by a sheaf complex of smaller amplitude). (v) follows from (iv), and the others follow from the fact that nearby cycles commute with proper pushforward, smooth pullback and box products. □

We will prove the commutativity constraint in the next section thanks to the following technical lemma. For this statement, we will need to assume that k is an algebraically closed field, so we assume so for the remainder of this note.

Lemma 13. *Let $f : X\to S$ be a morphism, D a smooth effective divisor on X giving immersions $i : Z\to X$, $j : X-Z\to X$. Suppose f is ULA with respect to some $\mathcal{F}\in D_c^b(X)$ and that $j^*\mathcal{F}$ is perverse. Then, $i^*\mathcal{F}[-1] = i^!\mathcal{F}[1]$ is perverse, and*

$$\mathcal{F} = j_{!*}j^*\mathcal{F}$$

is perverse.

Proof. This statement is local in the étale topology, so we may pick a smooth local equation $g : S \to \mathbb{A}^1$ for D and assume that $D = g^{-1}(0)$. Since g is smooth, a theorem of Illusie [SGA4.5, Leemme 2.14] implies that $\mathcal{F}$ is also ULA with respect to $gf : X \to \mathbb{A}^1$: this way we have reduced to the case where $S = \mathbb{A}^1$ is 1-dimensional. The upshot is that $Z = f^{-1}(0)$ is a fiber of f so we may use the theory of nearby and vanishing cycles. Lemma 11 shows that $i^*\mathcal{F}[-1] = \Psi(j^*\mathcal{F})[-1]$ is perverse (since vanishing cycles preserve perversity); equivalently, $\Phi(\mathcal{F}) = 0$ where the vanishing cycles Φ are defined by the triangle

$$i^*\mathcal{F} \to \Psi(j^*\mathcal{F}) \to \Phi(\mathcal{F}) \to .$$

The dual triangle (recall that nearby cycles commute with duality up to a twist, and the same goes for vanishing cycles up to an "Iwasawa" twist τ by theorems of Gabber and Beilinson) gives

$$\tau^{-1}\Phi(\mathcal{F}) \to \Psi(j^*\mathcal{F})[-2] \to i^!\mathcal{F} \to$$

so that $i^!\mathcal{F}[1] = \Psi(j^*\mathcal{F})[-1] = i^*\mathcal{F}[-1]$. Now, since $j_!$ is right t-exact and i_* is t-exact, the triangle

$$j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to$$

shows that $\mathcal{F}$ is concentrated in nonnegative degreees and that we have an exact sequence

$${}^{\mathfrak{p}}\mathcal{H}^0(j_!j^*\mathcal{F}) \to {}^{\mathfrak{p}}\mathcal{H}^0(\mathcal{F}) \to 0.$$

Dually, the triangle

$$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \to$$

shows that $\mathcal{F}$ is concentrated in nonpositive degreees and that we have an exact sequence

$$0 \to {}^{\mathfrak{p}}\mathcal{H}^0(\mathcal{F}) \to {}^{\mathfrak{p}}\mathcal{H}^0(j_*j^*\mathcal{F}).$$

Combined together, this implies that $\mathcal{F} = {}^{\mathfrak{p}}\mathcal{H}^0(\mathcal{F})$ is perverse and that

$$\mathcal{F} = \mathrm{Im}({}^{\mathfrak{p}}\mathcal{H}^0(j_!j^*\mathcal{F}) \to {}^{\mathfrak{p}}\mathcal{H}^0(j_*j^*\mathcal{F})) =: j_{!*}j^*\mathcal{F}.$$

□

5. The commutativity constraint

Let p denote the projection $\mathrm{Gr}_{G,X} = \mathrm{Gr}_G \times X \to \mathrm{Gr}_G$. This gives a faithful (since p has a section) embedding $p^*[1] : \mathrm{Sat} \to \mathrm{Sat}_X$: since p is smooth of relative dimension 1, the shift ensures it preserves perversity, and it lands into ULA sheaves since Sat consists of ULA perverse sheaves relatively to $\mathrm{Gr}_G \to \mathrm{pt}$ (by Remark 9, although it is elementary in this case). We will identify elements of Sat with elements of Sat_X via this embedding.

We are ready to prove the main theorem. We let U denote the open set $X^2 - \Delta X$ in X^2 and j the open immersion $\mathrm{Gr}_{G,X^2 \times_{X^2}} U \to \mathrm{Gr}_{G,X^2}$.

Theorem 14. *Let* $\mathcal{F}, \mathcal{G} \in \mathrm{Sat}_X$. *Then,*

$$\mathcal{F} * \mathcal{G} \simeq j_{!*}j^*(\mathcal{F} \boxtimes \mathcal{G}) \in \mathrm{Sat}_{X^2}\,.$$

Proof. Indeed, outside the diagonal ΔX, $\mathcal{F} * \mathcal{G}$ is simply the box product $\mathcal{F} \boxtimes \mathcal{G}$: via the isomorphism of Lemma 6, m is the identity outside the diagonal and the twisted product of $\mathrm{Gr}_{G,X}$ is just the usual product $\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}$ (this is analogous to the isomorphism $\mathrm{Gr}_G \tilde{\mathrm{Gr}}_G \simeq \mathrm{Gr}_G \times \mathrm{Gr}_G$). This is perverse, and ULA by Lemma 12. Thus, Lemma 13 implies that

$$\mathcal{F} * \mathcal{G} \simeq j_{!*}j^*(\mathcal{F} \boxtimes \mathcal{G}).$$

$\square$

Remark 15. This is slightly abusive: by $j^*(\mathcal{F} \boxtimes \mathcal{G})$ we mean the restriction of the box product from $\mathrm{Gr}^2_{G,X}$ to $\mathrm{Gr}^2_{G,X} \times_{X^2} U = \mathrm{Gr}_{G,X^2} \times_{X^2} U$, so this j is not the open immersion from this to Gr_{G,X^2} but instead the open immersion with same source but target $\mathrm{Gr}^2_{G,X}$.

This is the fusion product of $\mathcal{F}$ and $\mathcal{G}$. The upshot is that this has an obvious symmetry proving it is commutative! It also gives another proof that convolution preserves perverse sheaves without using the fact that the convolution map m is stratified semismall.

Corollary 16. *The convolution of two perverse sheaves* $\mathcal{F}, \mathcal{G} \in \mathrm{Sat} = \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$ *again lands in* Sat. *Moreover, we have a canonical isomorphism*

$$c_{\mathcal{F},\mathcal{G}} : \mathcal{F} \star \mathcal{G} \simeq \mathcal{G} \star \mathcal{F}.$$

Proof. By Lemma 7, the embedding $p^*[1] : \mathrm{Sat} \to \mathrm{Sat}_X$ is compatible with convolution:

$$\mathcal{F} \star \mathcal{G} = i^*[-1](\mathcal{F} * \mathcal{G}).$$

Let i denote the diagonal inclusion of $\mathrm{Gr}_{G,X}$ into Gr_{G,X^2}. The above expression shows that $\mathcal{F} \star \mathcal{G}$ is perverse by Lemma 13. To see that convolution is commutative, consider the involution sw on Gr_{G,X^2} which swaps x_1 and x_2. On the diagonal it doesn't do anything (i.e. $\mathrm{sw} \circ i = i$), so

$$\begin{aligned} i^*(\mathcal{F} * \mathcal{G}) &= i^* j_{!*} j^*(\mathcal{F} \boxtimes \mathcal{G}) \\ &= i^* \mathrm{sw}^* j_{!*} j^*(\mathcal{F} \boxtimes \mathcal{G}) \\ &= i^* j_{!*} j^* \mathrm{sw}^*(\mathcal{F} \boxtimes \mathcal{G}) \\ &= i^* j_{!*} j^*(\mathcal{G} \boxtimes \mathcal{F}) \\ &= i^*(\mathcal{G} * \mathcal{F}) \end{aligned}$$

since sw swaps box products (it swaps the two projections $\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X} \to \mathrm{Gr}_{G,X}$). $\square$

Remark 17. There is another way to construct convolution on Sat_X using nearby cycles which is more commonly called "fusion products". The Beilinson-Drinfeld Grassmanian Gr_{G,X^2} gives a fibration over $X^2 \simeq \mathbb{A}^1 \times \mathbb{A}^1$ with special fiber at $(0,0)$ isomorphic to Gr_G, and generic fiber at the generic point η isomorphic to $\mathrm{Gr}_G \times \mathrm{Gr}_G$. Then, as we saw in the proof of Lemma 13, the ULA condition implies that

$$\mathcal{F} \star \mathcal{G} = i[-1]^*(\mathcal{F} * \mathcal{G}) = \Psi(\mathcal{F}|_\eta \boxtimes \mathcal{G}|_\eta)[-1]$$

is the nearby cycles of $\mathcal{F} \boxtimes \mathcal{G}$ with respect to this fibration. We see that this is symmetric as before by pulling back via the automorphism sw (which is smooth so commutes with nearby cycles).

As another corollary, we are also able to prove that the fiber functor of Sat is monoidal (which we expect since we want to prove Sat is the category of representations of the Langlands dual of G). The symmetry however is a bit more involved, as the isomorphism $c_{\mathcal{F},\mathcal{G}}$ of Corollary 16 needs to be changed by a sign, see [Zhu, (5.2.8)].

Corollary 18. *The hypercohomology functor* $\mathbf{H} : \mathrm{Sat} \to \mathrm{Vect}^{\mathrm{gr}}$ *is monoidal.*

Proof. It suffice to prove this for Sat_X, since Sat embeds into Sat_X. Let q be the projection $\mathrm{Gr}_{G,X^2} \to X^2$. We factorise $\mathbf{H}$ as $i_{x*}\mathcal{H}$ where $x \in X^2$ is arbitrary and $\mathscr{H}$ is the functor

$$\mathcal{F} \mapsto \bigoplus_i \mathscr{H}^i(q_*\mathcal{F}) \in D^b_c(X^2).$$

Since i_{x*} is monoidal, it suffices to prove that $\mathscr{H}$ is also monoidal. Observe that q is smooth so by Lemma 12 (ii), $q_*\mathcal{F}$ is ULA with respect to the identity map $X^2 \to X^2$. Thus, by Lemma 12 (i), $\mathscr{H}$ lands into graded local systems. This implies that it suffices to prove that $\mathscr{H}$ is monoidal on a dense open subset of Gr_{G,X^2}. This is clear outside the diagonal: convolution is just box products and the results follows from Künneth's formula. ☐

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