ARKHIPOB–BEZRUKAVNIKOV EQUIVALENCE, PART 1: CONSTRUCTION OF THE FUNCTOR

Contents

1. Setup and motivation

Let me be a little quick with the motivation, since it was discussed in more detail in the past talks. In particular I don't want to dwell too much on any point, and instead just give an outline of the story we're trying to categorify.

Fix G a reductive group over $\bar{\mathbb{F}}_p$ with a Borel B. Set $F = \bar{\mathbb{F}}_p((t))$. The purpose of this seminar is to categorify the *local Langlands correspondence.*

Set $F = \bar{\mathbb{F}}_p((t)) \supseteq \bar{\mathbb{F}}_p[[t]] = \mathcal{O}$. This is not quite a local field, but it will be important for us later on to be working over $\bar{\mathbb{F}}_p$, so to avoid confusion I want to be working over it the entire time.

Fix a prime $\ell \neq p$. Recall that, by reversing the root datum of G, we can define a group $\check{G}/\bar{\mathbb{Q}}_\ell$, the *Langlands dual* of G.

The local Langlands conjecture, very roughly, asks if we can describe the irreducible representations of $G(F)$ via $\check{G}$. The point of this seminar is to categorify the local Langlands conjecture.

Date: May 4, 2026.

1.1. **The unramified case.** The first case of the local Langlands conjecture is the *unramified local Langlands conjecture*. This says that semisimple conjugacy classes in $\check{G}(\bar{\mathbb{Q}}_\ell)$ are in bijection with irreducible unramified representations of $G(F)$. This follows formally from the *Satake isomorphism*,

$$\bar{\mathbb{Q}}_\ell[G(\mathcal{O})\backslash G(F)/G(\mathcal{O})] \simeq K(\mathrm{Rep}(\check{G})) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}_\ell.$$

The first part of this seminar was on the *geometric Satake equivalence*. This was a symmetric monoidal equivalence of categories

$$\mathrm{Perv}(L^+G\backslash LG/L^+G) \simeq \mathrm{Rep}(\check{G}),$$

categorifying the Satake isomorphism.

Here, the right hand side is a category of coherent sheaves on $B\check{G}$, a stack over $\bar{\mathbb{Q}}_\ell$, and the left hand side is a category of certain ℓ-adic on something over $\bar{\mathbb{F}}_p$. Thus both sides are $\bar{\mathbb{Q}}_\ell$-linear, which is why this comparison makes sense.

1.2. **The tamely ramified case.** Once the unramified local Langlands conjecture is categorified, its natural to ask about the tamely ramified local Langlands conjecture (also called the Deligne–Langlands correspondence). Let $I \subseteq G(\mathcal{O})$ denote the Iwahori subgroup. Recall that we can also view I as a subgroup of the positive loop space L^+G.

As Kenta mentioned in the first talk, the tamely ramified local Langlands conjecture follows from the *Kazhdan–Lusztig isomorphism*, in the same way that the unramified local Langlands conjecture follows from the Satake isomorphism.

In the Kazhdan–Lusztig isomorphism, the spherical Hecke algebra is replaced by the Iwahori-Hecke algebra

$$\mathcal{H}_I = \bar{\mathbb{Q}}_\ell[I\backslash G(F)/I].$$

The replacement for $K(\mathrm{Rep}(\check{G}))$ is a little more subtle; let me recall it from Mingjia's talk.

Set $\check{\mathcal{N}}$ the nilpotent cone of $\check{\mathfrak{g}}$, the Lie algebra of $\check{G}$. This $\check{\mathcal{N}}$ is a singular variety; it admits a resolution of singularities

$$\check{\mathcal{N}}_{\mathrm{Spr}} \to \check{\mathcal{N}},$$

where

$$\check{\mathcal{N}}_{\mathrm{Spr}} := \{(x \in \check{\mathcal{N}}, \mathfrak{b} \text{ a Borel subalgebra}) \mid x \in \mathfrak{b}\}.$$

Example 1.1. Suppose $\check{G} = \mathrm{SL}_2$. A matrix $x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ in $\mathfrak{sl}_2$ is nilpotent if and only if $a^2 + bc = 0$. This is a cone in 3-space; in particular, it is singular at the origin.

Every Borel subalgebra of $\mathfrak{sl}_2$ contains a nonzero nilpotent matrix, and this nilpotent matrix is unique up to scaling. In particular, for each *nonzero* $x \in \check{\mathcal{N}}$, there is a unique Borel containing x. Thus $\check{\mathcal{N}}_{\mathrm{Spr}} \to \check{\mathcal{N}}$ is a bijection away from $x \neq 0$. At $x = 0$, the fiber is $\mathbb{P}^1$, as we can choose any Borel sub-algebra, which is equivalent to choosing a line in the nilpotent cone.

The *Steinberg* is defined as

$$\mathrm{St}_{\check{G}} = \check{\mathcal{N}}_{\mathrm{Spr}} \times_{\check{\mathcal{N}}} \check{\mathcal{N}}_{\mathrm{Spr}}.$$

The Kahzdan–Lusztig isomorphism, proven for us in Mingjia's talk, asserts that

$$\mathcal{H}_I \simeq K^{\check{G}\times\mathbb{G}_m}(\mathrm{St}_{\check{G}}) \otimes_{\mathbb{Z}[v^{\pm 1}]} \bar{\mathbb{Q}}_\ell.$$

Bezrukavnikov categorified this, proving Bezrukavnikov's equivalence

$$D^b_{\mathrm{cons}}(I\backslash LG/I) \simeq D^{b,\check{G}}_{\mathrm{coh}}(\mathrm{St}). \tag{1.2}$$

Warning 1.3. For (1.2) to be true, one needs to interpret St as a *derived* scheme. We will not do this in the rest of this talk, as we will only be proving the weaker Arkhipov–Bezrukavnikov theorem, where this subtlety doesn't appear.

This talk is the first in a series of three in which we prove the *Arkhipov–Bezrukavnikov* equivalence, a weaker form of (1.2). This weaker form plays a key role in the proof of Bezrukavnikov's equivalence.

1.3. **How did Mingjia prove Bezrukavnikov's equivalence?** To get to the Arkhipov–Bezrukavnikov equivalence, we first remind you of the structure of Mingjia's proof of the Kahzdan–Lusztig isomorphism.

She started by constructing a certain module M_{asph} over $\mathcal{H}_I$, called the *antispherical module*, given as the quotient $\mathcal{H}_I/(\underline{H}_w : w \in W_{\mathrm{ext}} \setminus W)$. Here the $\underline{H}_w$ are the Kahzdan–Lusztig basis elements of the Iwahori-Hecke algebra, which Naomi discussed in her talk, but you can forget that for now.[1]

She then showed that $\mathcal{H}_I$ acted faithfully on this antispherical module.

Finally, she showed that $K^{\check{G}\times\mathbb{G}_m}(\check{\mathcal{N}}_{\mathrm{Spr}})$ was a $K^{\check{G}\times\mathbb{G}_m}(\mathrm{St})$-module, and that $K^{\check{G}\times\mathbb{G}_m}(\mathrm{St})$ acted faithfully on it. By comparing $K^{\check{G}\times\mathbb{G}_m}(\check{\mathcal{N}}_{\mathrm{Spr}})$ and M_{asph}, she was able to deduce the Kahzdan–Lusztig isomorphism.

Bezrukavnikov's equivalence categorifies the Kahzdan–Lusztig isomorphism. The Arkhipov–Bezrukavnikov equivalence categorifies the comparison between $K^{\check{G}\times\mathbb{G}_m}(\check{\mathcal{N}}_{\mathrm{Spr}})$ and M_{asph}. In particular, the Arkhipov–Bezrukavnikov equivalence is a key step in the proof of the Bezrukavnikov equivalence.

1.4. **So, what is Arkhipov–Bezrukavnikov?** Let's now state the Arkhipov–Bezrukavnikov equivalence.

Recall that the affine flag variety $\mathrm{Fl} := L^+G/I$ has a decomposition into *Schubert cells* Fl_w, indexed by elements of W_{ext}, the *extended affine Weyl group*; this extended affine Weyl group fits into an exact sequence of groups

$$1 \to X_\bullet(G) \to W_{\mathrm{ext}} \to W \to 1.$$

Here, $X_\bullet(G)$ is the group of cocharacters of G, or equivalently characters of $\check{G}$. Let $j_w : \mathrm{Fl}_w \to \mathrm{Fl}$ denote the locally closed immersion of the corresponding Schubert cell. The analogue of $\mathbb{H}$ is the category $\mathrm{Perv}_I := \mathrm{Perv}^I(\mathrm{Fl})$ of I-equivariant perverse sheaves on Fl, and Naomi's elements $\underline{H_w}$ correspond to the perverse sheaves

$$L_w := j_{w,!*}(\bar{\mathbb{Q}}_\ell[\ell(w)]).$$

Remark 1.4. Recall $\mathrm{Fl}_w \simeq \mathbb{A}^{\ell(w)}$, which is why we need that cohomological shift.

We define

$${}^f\,\mathrm{Perv}_I := \mathrm{Perv}_I \,/\, \langle L_w \mid w \in W_{\mathrm{ext}} \setminus W_f \rangle \,.$$

We can now state the Arkhipov-Bezrukavnikov equivalence.

Theorem 1.5 (Arkhipov-Bezrukavnikov equivalence)**.** *There is a monoidal equivalence of categories*

$$D^{b,\check{G}}_{\mathrm{coh}}(\check{\mathcal{N}}_{\mathrm{Spr}}) \simeq D^b({}^f\,\mathrm{Perv}_I).$$

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[1] Technically Mingjia did her proof with the spherical module but let's pretend she used the antispherical module instead, since she could have and since it fits better into this story.

Remark 1.6 (Where did $\mathbb{G}_m$ go?)**.** In our motivation, there was a $\check{G} \times \mathbb{G}_m$-equivariance, and now we only have $\check{G}$-equivariance. What happened? Arkhipov-Bezrukavnikov assert in their introduction that a description of the $\check{G} \times \mathbb{G}_m$-equivariant coherent sheaves on the Springer is similar, but they only elaborate briefly (see their Remark 2). We will not say anything more about this.

The goal in today's talk is to construct a functor between these two categories; in two weeks, Kenta will prove our functor is an equivalence.

2. Coherent sheaves on the Springer resolution

We are going to be comparing some category of coherent sheaves to some category of perverse sheaves. It therefore behooves us to start by understanding both sides better.

2.1. **Some constructions we've already made.**

Let me just remind us that $\check{G}$ acts on $\tilde{\mathcal{N}}$ by

$$g \cdot (x, \mathfrak{b}) = (\mathrm{Ad}_g(x), \mathrm{Ad}_g(\mathfrak{b})).$$

The structure map $\check{\mathcal{N}}_{\mathrm{Spr}} \to \mathrm{pt}$ gives rise to a map $\check{G} \backslash \check{\mathcal{N}} \to \check{G} \backslash \mathrm{pt}$, and pulling back along this map gives us a monoidal functor

$$\mathrm{Rep}(\check{G}) \to \mathrm{Coh}^{\check{G}}(\check{\mathcal{N}}_{\mathrm{Spr}}),$$

which we denote by $V \mapsto \mathcal{O} \boxtimes V$.

Mingjia (implicitly) constructed certain interesting line bundles on $\check{G} \backslash \check{\mathcal{N}}_{\mathrm{Spr}}$. We now make explicit this construction.

Recall $\check{G}/\check{B}$ is the full flag variety, which is the moduli space of Borel subalgebras of the Lie algebra; the point $g\check{B}$ corresponds to the Borel subgroup $g\check{B}g^{-1}$. The Borel-Weil-Bott theorem gives, for every 1-dimensional representation ν of $\check{T}$, a $\check{G}$-equivariant line bundle $\mathcal{O}(\nu)$ on $\check{G}/\check{B}$, explicitly constructed as the bundle whose total space is $\check{G} \times_{\check{B}} \bar{\mathbb{Q}}_{\ell\nu}$, where we mean the quotient of $\check{G} \times \bar{\mathbb{Q}}_\ell$ by the B-action

$$(g, t) \cdot b := (gb, b^{-1}t).$$

Here, to interpret $b^{-1}t$ we use that the map $\check{B} \to \check{T} = \check{B}/\check{U}$ allows us to pullback ν to a representation of $\check{B}$.

There is a natural map

$$\check{\mathcal{N}}_{\mathrm{Spr}} \to \check{G}/\check{B},$$

$$(x, \mathfrak{b}) \mapsto \mathfrak{b},$$

which is $\check{G}$-equivariant.

These line bundles $\mathcal{O}(\nu)$ on $\check{G}/\check{B}$ can therefore be pulled back to get $\check{G}$-equivariant line bundles $\mathcal{O}(\nu)$ on $\check{\mathcal{N}}_{\mathrm{Spr}}$. This gives us a symmetric monoidal functor

$$\mathrm{Rep}(\check{T}) \to \mathrm{Coh}^{\check{G}}(\check{\mathcal{N}}_{\mathrm{Spr}}), \nu \mapsto \mathcal{O}(\nu).$$

We now have two types of $\check{G}$-equivariant sheaves on the Springer: these line bundles $\mathcal{O}(\nu)$, and the vector bundles $\mathcal{O} \boxtimes V$.

2.2. Interlude: Serre's trick and the base affine space.

Remark 2.1 (Recollection of Serre)**.** Let me first recall Serre's description of coherent sheaves on $\mathbb{P}^n$, because Arkhipov–Bezrukavnikov cite this as an inspiration for their argument so maybe it's good to have it loaded into our memories.

We can write $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m$, and in particular $\mathrm{Coh}(\mathbb{P}^n)$ can be viewed as the category of $\mathbb{G}_m$-equivariant sheaves on $\mathbb{A}^{n+1} \setminus \{0\}$. Coherent sheaves on $\mathbb{A}^{n+1} \setminus \{0\}$ are a little annoying to work with, but $\mathbb{A}^{n+1} \setminus \{0\}$ is very close to the affine scheme $\mathbb{A}^{n+1}$, which has a very easy to understand category of coherent sheaves.

There is a natural restriction functor

$$j^* : \mathrm{Coh}^{\mathbb{G}_m}(\mathbb{A}^{n+1}) \to \mathrm{Coh}^{\mathbb{G}_m}(\mathbb{A}^{n+1} \setminus \{0\}) \simeq \mathrm{Coh}(\mathbb{P}^n).$$

Serre proved that this functor realizes $\mathrm{Coh}(\mathbb{P}^n)$ as the Serre quotient of abelian categories

$$\mathrm{Coh}^{\mathbb{G}_m}(\mathbb{A}^{n+1}) / \mathrm{Coh}^{\mathbb{G}_m}_{\{0\}}(\mathbb{A}^{n+1}),$$

where we quotient by the category of coherent sheaves which are supported at the origin.

Let me recall that the proof of this boils down to showing essential surjectivity of: that every $\mathbb{G}_m$-equivariant coherent sheaf on $\mathbb{A}^{n+1} \setminus \{0\}$ extends to one on $\mathbb{A}^{n+1}$. This is done by observing that on Noetherian schemes, we can always extend coherent sheaves, and then do some averaging trick to extend the equivariant structure. The averaging tricks, at least in the generality I know how to do them, use that $\mathbb{G}_m$ is affine – we re-interpret equivariant structures as comodule structures: the pushforward $j_*\mathcal{E}$ will be $\mathbb{G}_m$-equivariant, but in general only quasi-coherent; we then find some subsheaf of it which is stable under the comultiplication and whose restriction is all of $\mathcal{E}$. The same argument works whenever we have an affine group scheme acting on some Noetherian scheme X in a way which preserves an open subscheme U.

Definition 2.2. Recall a scheme is *quasi-affine* if it is a quasicompact open subscheme of an affine scheme.

I know two things about geometric representation theory. One is Bernstein's $\mathcal{D}$-affineness of flag varieties. The other is, vaguely, how to construct this functor. Both things use in a crucial way that $\check{G}/\check{U}$ is quasi-affine, so it seems likely to me that this fact is important; let me elaborate on it.

First, recall that $\check{U} \subseteq \check{B}$ is the unipotent radical. The quotient $\check{G}/\check{U}$ is the *base affine space.* It is very close to $\check{G}/\check{B}$, which is ultimately why it will enter our proof later.

Theorem 2.3. *The base affine space is always quasi-affine. (Here we are using that our coefficients $\bar{\mathbb{Q}}_\ell$ are characteristic 0 and algebraically closed. I am not sure the strongest generality in which this quasi-affineness holds.)*

Proof. The name of the game is constructing lots of functions on $\check{G}/\check{U}$, or equivalently constructing lots of $\check{U}$-invariant functions on the affine group scheme $\check{G}$. I will show you where these functions are, then leave quasi-affineness as an exercise.

By Peter-Weyl, as a $\check{G} \times \check{G}$-module, we can describe the ring of functions on $\check{G}$ as

$$\mathcal{O}(\check{G}) = \bigoplus_{\nu} V_\nu \otimes V_\nu^*,$$

where V_ν is the irrep of highest weight ν. We remark that a function $v_\nu \otimes f_\nu \in V_\nu \otimes V_\nu^*$ acts on $\check{G}$ as

$$g \mapsto f_\nu(g \cdot v_\nu).$$

As the right action of $\check{U}$ on $\check{G}$ is free, we have

$$\mathcal{O}(\check{G}/\check{U}) \simeq \mathcal{O}(\check{G})^{\check{U}} = \bigoplus_{\nu} V_\nu \otimes (V_\nu^*)^{\check{U}}.$$

Highest weight theory, done concretely for SL_2 in Remark 2.4, tells us that $(V_\nu^*)^{\check{U}}$ is 1-dimensional, and an explicit nonzero element being projection onto the lowest weight space (which is one dimensional); call this projection $f_\nu : V_\nu \to \bar{\mathbb{Q}}_\ell$.

In particular,

$$\mathcal{O}(\check{G}/\check{U}) \simeq \bigoplus_{\nu} V_\nu \otimes (\bar{\mathbb{Q}}_\ell)_\nu,$$

where the subscript ν on the right factor indicates how T acts on it. □

Remark 2.4 (Highest weight theory for SL_2)**.** Suppose $\check{G} = \mathrm{SL}_2$. Then the unipotent radical consists of all the matrices $u_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. The dominant weights for $\check{G}$ are positive integers, and we can model the irrep of highest weight n as the space of homogeneous degree n polynomials in two variables x, y, with SL_2 acting via linear changes of variables. The highest weight vector is x^n, with the weight spaces spanned by $x^n, x^{n-1}y, ..., xy^{n-1}, y^n$, with y^n being the lowest weight vector.

The matrix u_a acts as

$$u_a \cdot x^d y^{n-d} \mapsto x^d(ax+y)^{n-d} = a^{n-d} \cdot x^n + \binom{n-d}{1} a^{n-d-1} x^{n-1} y + \binom{n-d}{2} a^{n-d-2} x^{n-2} y + \cdots + \binom{n-d}{n-d} x^d y^{n-d}.$$

In particular, observe that u_a sends each vector to itself *plus* some vectors of highest weight. Thus the space of $\check{U}$-coinvariants of V_n is 1-dimensional; in the quotient $(V_n)_{\check{U}}$, only the lowest weight vector y^n survives.

2.3. Implementing Serre's trick on the Springer.

We're trying to describe coherent sheaves on $\check{G} \backslash \check{\mathcal{N}}_{\mathrm{Spr}}$. Unfortunately, in general the Springer resolution is not quasi-affine, so we cannot immediately apply Serre's trick.

Recall that $\check{\mathcal{N}}$ embeds into $\check{\mathfrak{g}} \times \check{G}/\check{B}$. Now you probably see the point: we're going to cover this by $\check{\mathfrak{g}} \times \check{G}/\check{U}$. Indeed, we define $\check{\mathcal{N}}^{\mathrm{qaf}}_{\mathrm{Spr}}$ to be the pre-image of $\check{\mathcal{N}}_{\mathrm{Spr}} \hookrightarrow \mathfrak{g} \times \check{G}/\check{B}$ along the map $\mathfrak{g} \times \check{G}/\check{U}$. As $\check{G}/\check{U} \to \check{G}/\check{B}$ is a $\check{T} = \check{B}/\check{U}$-torsor, this $\check{\mathcal{N}}^{\mathrm{qaf}}_{\mathrm{Spr}}$ is a $\check{T}$-torsor over the Springer resolution. This $\check{\mathcal{M}}^{\mathrm{qaf}}_{\mathrm{Spr}}$ is quasi-affine (as we'll check below), with a natural action of $\check{G} \times \check{T}$.

Lemma 2.5. *The map*

$$\check{\mathcal{N}}^{\mathrm{qaf}}_{\mathrm{Spr}} \to \check{\mathcal{N}}_{\mathrm{Spr}}$$

induces an isomorphism of stacks

$$(\check{G} \times \check{T}) \backslash \check{\mathcal{N}}^{\mathrm{qaf}}_{\mathrm{Spr}} \simeq \check{G} \backslash \check{\mathcal{N}}_{\mathrm{Spr}}.$$

Proof. Easy to see by checking on points. □

So, instead of working with $\check{G}$-equivariant coherent sheaves on $\check{\mathcal{N}}_{\mathrm{Spr}}$, we can work with $\check{G} \times \check{T}$-equivariant sheaves on $\check{\mathcal{N}}^{\mathrm{qaf}}_{\mathrm{Spr}}$. Serre's trick allows us to understand these more easily.

As the Springer resolution is a closed subscheme of $\check{\mathfrak{g}} \times \check{G}/\check{B}$, it follows that $\check{\mathcal{N}}^{\mathrm{qaf}}_{\mathrm{Spr}}$ is a closed subscheme of $\check{\mathfrak{g}} \times \check{G}/\check{U}$. Now, $\check{\mathfrak{g}} \times \check{G}/\check{U}$ is an open subscheme of the affine scheme $\check{\mathfrak{g}} \times \operatorname{Spec} \mathcal{O}(\check{G}/\check{U})$, so $\check{\mathcal{N}}^{\mathrm{qaf}}_{\mathrm{Spr}}$ is a locally closed subscheme of an affine scheme.

Next, set $\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}$ the closed subscheme of $\check{\mathfrak{g}} \times \operatorname{Spec} \mathcal{O}(\check{G}/\check{U})$ given as follows.

Construction 2.1. Recall that the Springer consists of pairs $(x, \mathfrak{b}')$ of a Borel sub-algebra and some $x \in \check{\mathfrak{g}}$ which is nilpotent and lies in $\mathfrak{b}'$. Instead of Borel subalgebras, we could talk about Borel subgroups; this requires us to translate the condition.

A coset $gB \in G/B$ corresponds to a Borel subgroup gBg^{-1}. The Lie algebra of gBg^{-1} is the Lie subalgebra of $\check{\mathfrak{g}}$ consisting of all x so that $x \in \mathrm{Ad}(g)\check{\mathfrak{b}}$ (for $\check{\mathfrak{b}}$ the Lie algebra of the standard Borel); or equivalently, so that x vanishes in $T_{gB}(G/B) \simeq \mathfrak{g}/\mathrm{Ad}(g)\mathfrak{b}$.

Differentiating the natural action of G on G/U gives a map

$$a_g : \mathfrak{g} \to T_{gU}(G/U),$$

and we define a vector field V on $\check{\mathfrak{g}} \times \check{G}/\check{U}$ by asserting that

$$V(x, gU) := (0, a_g(x)).$$

This vector field V vanishes identically on $\check{\mathcal{N}}^{\mathrm{qaf}}_{\mathrm{Spr}}$, and in fact it vanishes exactly on this quasi-affine Springer.

The vector field V also defines a derivation on $\mathcal{O}(\check{\mathfrak{g}}) \times \mathcal{O}(\check{G}/\check{U})$, and the image of this derivation defines a closed subscheme of $\check{\mathfrak{g}} \times \operatorname{Spec} \mathcal{O}(\check{G}/\check{U})$. We call this closed subscheme $\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}$. Observe that

$$\check{\mathcal{N}}^{\mathrm{qaf}}_{\mathrm{Spr}} = \check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}} \cap (\check{\mathfrak{g}} \times \check{G}/\check{U}),$$

so that the quasi-affine Springer is an open subscheme of the affine Springer.

Remark 2.6. It will be useful for us later on to spell out explicitly the generators of the ideal of definition of $\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}$.

First, recall that we differentiated the action of $\check{G}$ on $\check{G}/\check{U}$ to get a map

$$a : \mathfrak{g} \to T(G/U).$$

This action sends $x \in \mathfrak{g}$ to the derivation

$$a(x) : \mathcal{O}(G/U) = \bigoplus V_\lambda \to \mathcal{O}(G/U) = \bigoplus V_\lambda$$

which comes as the direct sum of maps $V_\lambda \xrightarrow{x} V_\lambda$, using that the $\check{G}$-action on V_λ gives rise to an action of $\check{\mathfrak{g}}$ on V_λ by differentiation.

Hence the derivation of $\mathcal{O}(\check{\mathfrak{g}}) \times \mathcal{O}(\check{G}/\check{U})$ corresponding to the vector field V above is induced by the maps

$$d : \mathcal{O}(\mathfrak{g}) \otimes V_\lambda \to \mathcal{O}(\mathfrak{g}) \otimes V_\lambda$$

given as follows: choose $x_1, ..., x_n$ a basis of $\mathfrak{g}$, and then

$$d(f \otimes v) = \sum_{i=1}^{n} (f x_i^\vee) \otimes (x_i \cdot v).$$

3. Some geometry on the constructible side

We are going to need to know about some I-equivariant perverse sheaves on Fl. We already have one source of these: Gaitsgory's central functor

$$Z : \mathrm{Rep}(\check{G}) \to \mathrm{Perv}_I .$$

Xingzhu, in the previous talk, gave us a second source of perverse sheaves. In his talk, he proved that for any *dominant* weight ν, the constructible sheaf $Rj_{\nu,*}\bar{\mathbb{Q}}_\ell[\ell(\nu)]$ (he called it the costandard) was actually perverse. He extended this to a monoidal functor

$$J : \mathrm{Rep}(\check{T}) \to \mathrm{Perv}_I,$$

denoted by $\nu \mapsto J_\nu$, where for ν dominant we have $J_\nu := Rj_{\nu,*}\bar{\mathbb{Q}}_\ell[\ell(\nu)]$.

In his talk, Xingzhu also explored an interesting relationship between J and Z which we will need. He phrased it by asserting that Z factored through the subcategory of "Wakimoto filtered" perverse sheaves, and that the composition $\mathrm{gr} \circ Z : \mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{T})$ was the natural restriction functor.

We are going to use a certain part of Xingzhu's *proof* of that result, so let me state that part of the proof precisely.

Lemma 3.1. *Let ν be a dominant weight. Then, there is a canonical equivalence*

$$Rj_{\nu,*}j_\nu^* Z(V_\nu) \simeq J_\nu.$$

In fact, we even have the slightly stronger equivalence

$$j_\nu^* Z(V_\nu) \simeq \bar{\mathbb{Q}}_\ell[\ell(\nu)].$$

Remark 3.2. In Xingzhu's talk, the left hand side of the first formula appeared as the ν-piece of the associated graded of the Wakimoto filtration.

Proof. This affine flag variety is related to the affine Grassmannian $\mathrm{Gr} = G(F)/G(\mathcal{O})$ via a map

$$\pi : \mathrm{Fl} \to \mathrm{Gr} .$$

Moreover, $\pi_* \circ Z$ is just the geometric Satake equivalence, so that

$$\pi_* Z(V_\nu) \simeq \bar{j}_{\nu,!*}\bar{\mathbb{Q}}_\ell[\ell(\nu)],$$

for $\bar{j}_\nu : \mathrm{Gr}_\nu \hookrightarrow \mathrm{Gr}$ the inclusion of the Schubert cell.

So, our desired restriction formula is true on the affine Grassmannian. Arkhipov–Bezrukavnikov then give some argument I didn't understand to claim that was sufficient; this is there lemma 9, but it seems that the map $\pi : \mathrm{Fl}_\nu \to \mathrm{Gr}_\nu$ must be special in some way I cannot comprehend for them to reason as they do. ☐

4. Construction of F at the abelian level

4.1. Preamble: The idea of our construction.

We're trying to construct a functor

$$F : D^{b,\check{G}}_{\mathrm{coh}}(\check{\mathcal{N}}_{\mathrm{Spr}}) \to D^b(\mathrm{Perv}_I).$$

While the Arkhipov–Bezurkavnikov equivalence only holds at the derived level, we're going to construct the comparison functor at the abelian level. However, the abelian categories we use might surprise you!

4.2. **Step 0a: Simplifying coherent sheaves.** Define $\mathrm{Coh}_f^{\check{G}}(\check{\mathcal{N}}_{\mathrm{Spr}})$ the full subcategory of $\mathrm{Coh}^{\check{G}}(\check{\mathcal{N}}_{\mathrm{Spr}})$ whose objects are those of the form $\mathcal{O}(\nu) \boxtimes V$. This section will only construct a functor $\mathrm{Coh}_f^{\check{G}}(\check{\mathcal{N}}_{\mathrm{Spr}}) \to \mathrm{Perv}_I$, which is why we keep saying this is our first attempt at defining the comparison functor. In the next section, we'll finish. The advantage of working with this Coh_f category, though is that we know at least where objects of it should map to; the only thing left to do is to understand where morphisms go, and also understand the monoidal structure.

Recall that, in the previous section, we described $\check{G}$-equivariant coherent sheaves on $\check{\mathcal{N}}_{\mathrm{Spr}}$ as $\check{G} \times \check{T}$-equivariant coherent sheaves on $\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}$. In particular, we can compute hom's between these objects $\mathcal{O}(\nu) \boxtimes V$ by viewing them as modules over the spring $\mathcal{O}(\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}})$. We will use this to define how our functor acts on arrows.

4.3. **Step 0b: Simplifying perverse sheaves.** We now simplify the category of perverse sheaves. Let $\mathcal{C}$ denote the (non-full) subcategory of Perv_I where

(1) the objects are those in the image of $Z \times J : \mathrm{Rep}(\check{G} \times \check{T}) \to \mathrm{Perv}_I$,
(2) the morphisms those morphisms $\varphi : C \to C'$ in Perv_I such that, for any $C'' \in \mathrm{im}(Z \times J)$, the diagram

$$\begin{tikzcd}
C'' * C \arrow[r, "\mathrm{id} * \varphi"] \arrow[d] & C'' * C' \arrow[d] \\
C * C'' \arrow[r, "\varphi * \mathrm{id}"] & C' * C''
\end{tikzcd}$$

commutes, where the horizontal isomorphisms come from taking the image of the symmetry isomorphisms in the symmetric monoidal category $\mathrm{Rep}(\check{G} \times \check{T})$.

This category $\mathcal{C}$ is a symmetric monoidal subcategory of Perv_I. We can give an alternative, and very simple, description of $\mathcal{C}$.

First, recall that $\mathcal{O}(\check{G} \times \check{T})$ with its regular representation is an ind-object in $\mathrm{Rep}(\check{G} \times \check{T})$. Define

$$A := \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\mathbf{1}_{\mathcal{C}}, (Z \times J)(\mathcal{O}(\check{G} \times \check{T}))) = \bigoplus_V V^* \otimes \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, F(V)),$$

where the sum ranges over all irreps of V of $\check{G} \times \check{T}$.

This A can be viewed as a commutative $\bar{\mathbb{Q}}_\ell$-algebra. Indeed, given maps $a_1, a_2 : \mathbf{1}_{\mathcal{C}} \to (Z \times J)(\mathcal{O}(\check{G} \times \check{T}))$, we can define their product $a_1 a_2 \in A$ as the composition

$$\mathbf{1} = \mathbf{1} * \mathbf{1} \xrightarrow{a_1 * a_2} (Z \times J)(\mathcal{O}) * (Z \times J)(\mathcal{O}) = (Z \times J)(\mathcal{O} \otimes \mathcal{O}) \to (Z \times J)(\mathcal{O}),$$

where the last arrow is the mutliplication morphism.

The right $\check{G} \times \check{T}$-module structure on $\mathcal{O}(\check{G} \times \check{T})$ gives rise to an action of $\check{G} \times \check{T}$ on A. We define

$$\mathrm{Mod}_A^{\check{G} \times \check{T}, f}$$

the full subcategory of $\check{G} \times \check{T}$-equivariant A-modules whose objects are of the form $V \otimes A$ for $V \in \mathrm{Rep}(\check{G} \times \check{T})$.

Proposition 4.1. *There is an equivalence of symmetric monoidal categories*

$$\mathcal{C} \simeq \mathrm{Mod}_A^{\check{G} \times \check{T}, f},$$

sending $(Z \times J)(V) \in \mathcal{C}$ *to* $V \otimes A$.

Proof. This is mostly a matter of unwinding definitions; see [1], proposition 6.3.5. □

We now construct a morphism of $\bar{\mathbb{Q}}_\ell$-algebras

$$\mathcal{O}(\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}) \to A \tag{4.2}$$

which is $\check{G} \times \check{T}$-equivariant. This gives rise to a monoidal extension of scalars functor

$$\mathrm{Mod}_f^{\check{G}\times\check{T}}(\mathcal{O}(\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}})) \to \mathrm{Mod}_f^{\check{G}\times\check{T}}(A) \simeq \mathcal{C},$$

which through composition then gives a monoidal functor

$$\tilde{F} : \mathrm{Coh}_f^{\check{G}}(\check{\mathcal{N}}_{\mathrm{Spr}}) \to \mathrm{Mod}_f^{\check{G}\times\check{T}}(\mathcal{O}(\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}})) \to \mathcal{C} \hookrightarrow \mathrm{Perv}_I\,.$$

Recall that $\mathcal{O}(\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}})$ is a quotient of $\mathcal{O}(\check{g}) \times \mathcal{O}(\check{G}/\check{U})$. This description will be how we define (4.2).

4.4. **Step 1: Mapping out of $\check{\mathfrak{g}}$.** This was essentially explained by Bhargav during his digression on endomorphisms. Gaitsgory's central functor gives us a symmetric monoidal functor

$$Z : \mathrm{Rep}(\check{G}) \to \mathcal{C}.$$

Since we are working over $\bar{\mathbb{F}}_p$ on the constructible side, we fix once and for all a trivialization of the Tate twist. Keeping this trivialization in mind, recall that Bhargav constructed for us (by taking the logarithm of monodromy) endomorphisms

$$n_V : Z(V) \to Z(V)$$

for every $V \in \mathrm{Rep}(\check{G})$, obeying

$$n_{V\otimes W} = n_V \otimes 1 + 1 \otimes n_W. \tag{4.3}$$

We now construct an element $x \in \check{\mathfrak{g}} \otimes A$ such that x acts on $Z(V) = V \otimes A$ as n_V. Here, recall that the action $\check{G} \to \mathrm{Aut}(V)$ can be differentiated to get $\check{\mathfrak{g}} \to \mathrm{End}(V)$, explaining how $\check{\mathfrak{g}} \otimes A$ acts on $V \otimes A$.

We construct x using the Tannakian formalism. Think of Z as a fiber functor to Mod_A . We have a natural transformation $n : Z \to Z$ obeying a certain condition; the Tannakian formalism identifies $\check{\mathfrak{g}} \otimes A \simeq \mathrm{End}^{\otimes}(Z)$.

Remark 4.4. Recall that, to us the Tannakian formalism identified a fiber functor $\omega : \mathcal{D} \to \mathrm{Mod}_A$ with a group scheme G such that

$$G(R) = \mathrm{Aut}^{\otimes}(\omega \otimes R).$$

We can view $\mathfrak{g}(R)$ as the kernel of the map $G(R[\epsilon]/(\epsilon^2)) \to G(R)$, and in this way identify endomorphisms of the fiber functor with points of the Lie algebra.

These endomorphisms n_V are $\check{G}$-equivariant, so we even have that $x \in (\check{\mathfrak{g}} \otimes A)^G$. Thus we get a $\check{G}$-equivariant algebra homomorphism

$$\mathcal{O}(\check{\mathfrak{g}}) \to A.$$

4.5. **Step 2: Mapping out of $\mathcal{O}(\check{G}/\check{U})$.** We now construct a $\check{G} \times \check{T}$-equivariant algebra homomorphism

$$\mathcal{O}(\check{G}/\check{U}) \to A.$$

As we already saw

$$\mathcal{O}(\check{G}/\check{U}) = \bigoplus_{\nu} V_\nu,$$

it suffices to construct $\check{G}$-equivariant maps

$$b_\nu : V_\nu \otimes A \to A(\nu)$$

obeying the *Plucker relations*: the maps

$$V_\nu \otimes V_\mu \otimes A \to A(\nu+\mu)$$

given by

$$V_\nu \otimes V_\mu \otimes A \xrightarrow{\mathrm{id}\otimes b_\mu} V_\nu \otimes A(\mu) \xrightarrow{b_\nu(\mu)} A(\nu+\mu)$$

and

$$V_\nu \otimes V_\mu \otimes A \xrightarrow{m_{\nu,\mu}} V_{\nu+\mu} \otimes A \xrightarrow{b_{\nu+\mu}} A(\nu+\mu)$$

coincide, where

$$m_{\nu,\mu} : V_\nu \otimes V_\mu \to V_{\nu+\mu}$$

is the multiplication on $\mathcal{O}(\check{G}\check{U})$.

Remark 4.5. Here, the fact that the output of b_ν is twisted by ν is what allows us to make our map $\check{T}$-equivariant.

We produce these b_ν via *highest weight arrows.* This will come from the Wakimoto filtration.

It is equivalent to produce an arrow $V_\nu \to \mathcal{O}(\nu)$ in $\mathcal{C}$. Thus we need to produce an arrow $Z(V_\nu) \to \mathcal{O}(\nu)$ in Perv_I, and then check some compatibilities to see our arrow belongs to $\mathcal{C}$.

Adjunction gives us a canonical map

$$Z(V_\nu) \to Rj_{\bar{\nu},*}j^*_{\bar{\nu}}Z(V_\nu),$$

and so by Lemma 3.1, we get an arrow

$$b_\nu : Z(V_\nu) \to J_\nu.$$

Proposition 4.6. *The highest weight arrows obey the Plucker relations: that is, there is a commutative diagram*

$$\begin{tikzcd}
Z(V_\nu) * Z(V_\mu) \arrow[r, "Z(m_{\nu,\mu})"] \arrow[d, "b_\nu * b_\mu"] & Z(V_{\nu+\mu}) \arrow[d, "b_{\nu+\mu}"] \\
J_\nu * J_\mu \arrow[r] & J_{\nu+\mu}
\end{tikzcd}$$

Here, the bottom map is an isomorphism coming from the fact that J is symmetric monoidal.

Proof. Recall that the highest weight arrows come from the unit of adjunction; in this adjunction, the left adjoint $j^*_{\bar{\nu}}$ is strict symmetric monoidal, and hence formation of units is symmetric monoidal, as desired. ☐

Remark 4.7. One should also check that the highest weight arrows belong in $\mathcal{C}$; this can be reduced to checking that, for any dominant weight λ, the diagram

$$\begin{tikzcd}
\mathcal{O}(\lambda) * Z(V_\nu) \arrow[r, "\mathrm{id} * b_\lambda"] \arrow[d] & \mathcal{O}(\lambda) * \mathrm{gr}_{\bar{\nu}} Z(V_\nu) \arrow[d] \\
Z(V_\nu) * \mathcal{O}(\lambda) \arrow[r, "b_\lambda * \mathrm{id}"] & \mathrm{gr}_{\bar{\nu}} Z(V_\nu) * \mathcal{O}(\lambda)
\end{tikzcd}$$

commutes.

It seems checking this compatibility is somewhat subtle; proposition 4.6.12 of [1] does it, but it takes them a few pages of computations and I'm unsure if there is any simplification. Arkhipov–Bezrukavnikov suggest it is obvious, but I don't see a way to avoid those computations from [1].

Thus we have our arrows b_ν, and hence a $\check{G}\times\check{T}$-equivariant algebra homomorphism

$$\mathcal{O}(\check{G}/\check{U}) \to A.$$

4.6. **Step 3: Putting it all together.** We now have a map $\mathcal{O}(\check{\mathfrak{g}}) \otimes \mathcal{O}(\check{G}/\check{U}) \to A$. We wish to show this morphism descends to $\mathcal{O}(\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}})$, which requires us being a little more explicit about how to find this quotient. The tautological vector field V introduced in

We now check that, for every dominant weight ν, we have $b_\nu \circ n_{V_\nu} = 0$. This is enough for the morphism to descend to $\mathcal{O}(\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}})$, which you can see by really unwinding definitions in the above, as done by [1].

Proposition 4.8. *We do indeed have* $b_\nu \circ n_{V_\nu} = 0$.

Proof. It suffices to check that $j_\nu^*(n_{V_\nu}) = 0$. Recall that n_{V_ν} is nilpotent; as

$$j_\nu^* Z(V_\nu) \simeq \bar{\mathbb{Q}}_\ell[\ell(\nu)],$$

any nilpotent endomorphism of it must vanish. □

5. Extending to the derived category

In the last section, we constructed a monoidal functor

$$\tilde{F} : \mathrm{Coh}^{\check{G}}_f(\check{\mathcal{N}}_{\mathrm{Spr}}) \to \mathrm{Mod}^{\check{G}\times\check{T}}_f(\mathcal{O}(\check{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}})) \to \mathcal{C} \hookrightarrow \mathrm{Perv}_I .$$

We now identify

$$D^b(\mathrm{Coh}^{\check{G}}_f(\check{\mathcal{N}}_{\mathrm{Spr}})) \simeq D^{b,\check{G}}_{\mathrm{coh}}(\check{\mathcal{N}}). \tag{5.1}$$

Lemma 5.2. *We have*

$$\mathrm{Coh}^{\check{G}}(\check{\mathcal{N}}_{\mathrm{Spr}}) \simeq \mathrm{Coh}^{\check{B}}(\check{\mathfrak{u}}).$$

Proof. Recall $\check{\mathcal{N}}_{\mathrm{Spr}} = \check{G} \times^{\check{B}} \check{\mathfrak{u}}$, and so we have an isomorphism of stacks

$$\check{G}\backslash\check{\mathcal{N}}_{\mathrm{Spr}} \simeq \check{B}\backslash\check{\mathfrak{u}}.$$

□

We now show that the line bundles $\mathcal{O}(\mu)$, with μ an *arbitrary weight* generate the triangulated category $D^b(\mathrm{Coh}^{\check{G}}(\mathcal{N}_{\mathrm{Spr}}))$. Then we will show that we can resolve $\mathcal{O}(\mu)$ by objects $V \boxtimes \mathcal{O}(\nu)$ where ν is dominant.

For the first step, note that any object $V \otimes \mathcal{O}(\check{\mathfrak{u}})$, with V a finite dimensional $\check{B}$-representation, has a filtration with subqutoients of the form $\mathcal{O}(\mu)$, and hence it suffices to check the following lemma.

Lemma 5.3. *Every object in* $\mathrm{Coh}^{\check{B}}(\check{\mathfrak{u}})$ *admits a resolution by objects* $V \otimes \mathcal{O}(\check{\mathfrak{u}})$.

Proof. Let M be a finite type $\check{B}$-equivariant $\mathcal{O}(\check{\mathfrak{u}})$-module. We can find some resolution

$$P^\bullet \to M \to 0$$

where each P^{-n} is of the form $V \otimes \mathcal{O}(\check{\mathfrak{u}})$, but the resolution may be of infinite length; we need to show it can be take nto have finite length.

Via the cocharacter 2ρ of $\check{T}$, we can regard any $\check{T}$-module (and hence any $\check{B}$-module) as a graded vector space.

In particular, we can treat M as a graded $\mathcal{O}(\check{\mathfrak{u}})$-module. But $\check{\mathfrak{u}} \simeq \mathbb{A}^{\dim \check{\mathfrak{u}}}$, so by the Hilbert syzygy theorem $\ker(P^{-\dim(\check{\mathfrak{u}})} \to P^{-\dim(\check{\mathfrak{u}})+1})$ is a finitely generated graded free module M'.

Now we induct on the rank of M' to show it has a resolution of the desired form. M' splits into pieces of different degrees; the smallest degree piece M'' also is stable under the $\check{B}$-action, as each graded piece is stable under $\check{T}$ and the lowest degree piece is also stable under $\check{U}$, as $\check{U}$ always acts by lowering degrees.

Thus we can split off a direct summand of M' in the category of $\check{B}$-equivariant $\mathcal{O}(\check{\mathfrak{u}})$-modules; iterating this process we conclude. □

Lemma 5.4. *Every line bundle $\mathcal{O}(\lambda)$ can be resolved by a complex of terms of the form $V \boxtimes \mathcal{O}(\nu)$ for ν dominant.*

Proof. Choose some dominant weight ν such that $\nu + \lambda$ is dominant. Recall that we constructed certain arrows

$$B_\nu : V_\nu \boxtimes \mathcal{O} \to \mathcal{O}(\nu).$$

Set $d = \dim V_\nu$. Koszul complex of this arrow is an acyclic complex

$$0 \to \wedge^d(V_\nu) \boxtimes \mathcal{O} \to \wedge^{d-1}(V_\nu) \boxtimes \mathcal{O}(\nu) \to \cdots \to V_\nu \boxtimes \mathcal{O}((d-1)\nu) \to \mathcal{O}(d\nu) \to 0.$$

Tensor with $\wedge^d(V_\nu)^* \boxtimes \mathcal{O}(\lambda)$ and we conclude. □

References

[1] Pramod N. Achar and Simon Riche. *Central sheaves on affine flag varieties.*

[2] Sergey Arkhipov and Roman Bezrukavnikov. *Perverse sheaves on affine flags and Langlands dual group.*