Coloring-Based Knot Invariants in Projective Space

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Abstract

In this paper, we will determine and characterize knot invariants related to coloring in $\mathbb{RP}^3$ and $\mathbb{RP}^2 \times (0,1)$. In $\mathbb{S}^3$, there exist notions of colorings related to coloring the regions and strands, and we extend those to $\mathbb{RP}^3$ and $\mathbb{RP}^2 \times (0,1)$. In $\mathbb{RP}^2 \times (0,1)$, we can establish an involutive function that determines the colors of antipodal points and attach it to the structure of a quandle. We attach such a function to the Joyce quandle, giving us a knot invariant that allows us to distinguish between two links. However, such a function does not work as well in $\mathbb{RP}^3$, so we instead establish a knot invariant from coloring regions of the projections of certain links, based on the Dehn presentation of the fundamental group of the knot complement. We establish a similar invariant for all links in $\mathbb{RP}^2 \times (0,1)$.

Summary

In this paper, we study knots as abstract mathematical objects on projective spaces. In particular, we will study knots that live in a sphere where diametrically opposite points are identified with, or"glued to" each other. We will also study knots that live in a cylinder where diametrically opposite points of the cross-sectional circles are identified with each other. We aim to distinguish between knots in these spaces by finding and characterizing methods of coloring these knots under a set of restraints, so that the number of colorings satisfying our restraints always stays the same no matter how the knot is deformed. This will tell us that any two knots with a different number of colorings must be different from each other. In one of the spaces, we found a way to color the arcs of the projection—the image seen when viewed from above—of the knot. In both spaces, we found a way to color the regions formed by the projection of the knot such that the number of colorings is always the same no matter how the knot is deformed.

1 Introduction

In the late nineteenth century, scientists such as Helmholtz, Tait, and Thomson believed that the chemical elements were made of knots, which are closed loops in space, and links, which are a disjoint union of knots, in a material they called "ether." This motivated the tabulation of knots, because they believed that they were making a table of the elements [1]. Although this theory was disproved, mathematicians continued to be interested in tabulating and identifying knots, due to their intrinsic interest. In particular, they wanted to be able to distinguish knots from each other.

Knots in $\mathbb{R}^3$ (which are equivalent to knots in $\mathbb{S}^3$) can be visualized diagramatically by a projection into $\mathbb{R}^2$, where over-crossings and under-crossings are distinguished from one another. This results in an image of a knot that is comprised of arcs. In 1926, Reidemeister established the three Reidemeister moves on these knot diagrams, which consist of twisting an arc, pulling one arc over or under another arc, or pulling an arc over or under a crossing. Furthermore, he proved that when such moves are combined with planar isotopies—deformations of the planar projection of the knot—they can transform any projection of a knot into another projection of the same knot [2].

These moves motivate the quandle, a system for coloring arcs of a projection of the knot such that the number of colorings is a knot invariant. The structure of the quandle follows axioms that respect the Reidemeister moves so that any Reidemeister move will not change the number of colorings of a knot under a quandle. For example, the Fox n-coloring colors arcs with elements of $\mathbb{Z}/n\mathbb{Z}$ such that at each crossing, the sum of the colors assigned to the arcs on both sides of the under-crossing is equal to two times the color of the over-crossing, as shown in Figure 1. Other types of coloring-based knot invariants include the Dehn coloring, which colors regions of the knot projection, and the Alexander-Briggs coloring, which colors crossings of the knot projection [3, 4]. In 1990, Drobotukhina began studying knots in $\mathbb{RP}^3$, establishing two slide moves for those knots in addition to the three Reidemeister moves, and found an analogue to the Jones polynomial for links in $\mathbb{RP}^3$ [5].

In this paper, we characterize knot invariants related to quandles in $\mathbb{RP}^2 \times (0,1)$ by extending the quandle to contain axioms that correspond to the slide moves as well, and find an example of a quandle that helps distinguish between two links in $\mathbb{RP}^2 \times (0,1)$ that have the same 2-covering. We also determine a region-based coloring invariant for links in $\mathbb{RP}^3$.

We begin in Section 2 by establishing the preliminary ideas and definitions needed to study knots in projective space. In Section 3, we introduce a structure that builds off of a quandle to help distinguish knots in $\mathbb{RP}^2 \times (0,1)$. In particular, in Theorem 3.12, we show

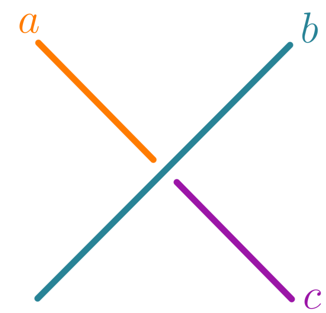

Figure 1: Rule for Fox n-coloring. Graphic created by the student researcher based on [3] using Ibis Paint X, 2025.

that for any quandle Q and involution φ such that $\varphi(a * b) = \varphi(a) \mathbin{\overline{*}} \varphi(b)$, the number of (Q, φ)-colorings of a link in $\mathbb{RP}^2 \times (0,1)$ is an oriented link invariant. We also extend the notion of the fundamental group of Fox colorings to a $\mathbb{Z}[t]/(t^2-1)$-module for such colorings. However, the structure of $\mathbb{RP}^3$ forces any quandle behind a similar system of coloring in $\mathbb{RP}^3$ to be trivial. Thus, in Section 4, we determine a way to color regions of the diagram of a knot in $\mathbb{RP}^3$ if its number of boundary points is divisible by 4 in Theorem 4.6, and also introduce a region-based coloring system for knot diagrams in $\mathbb{RP}^2 \times (0,1)$ in Theorem 4.7.

2 Preliminaries

First, we will define knots and links, since these are the objects that provide motivation for the structure of a quandle.

Definition 2.1. A *knot* is an oriented embedding of $\mathbb{S}^1$ into a 3-manifold. A *link* (a disjoint union of knots) is an embedding of $\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1$ into a 3-manifold.

By an abuse of notation, we often identify links with their images and view them as a subset of the manifold M that they are embedded in.

2.1 Real Projective Space

Because we are studying colorings of knots and links in $\mathbb{RP}^3$, we will now describe real projective space.

Definition 2.2. Let $\mathbb{RP}^n$ be the n-manifold $\mathbb{S}^n/\{\pm 1\}$, which is the n-sphere with antipodal points identified with each other. In other words, it can be described as the set

$$\{(x_0, x_1, \ldots, x_n) \mid x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}/\sim,$$

where the equivalence relation is given by $(x_0, x_1, \ldots, x_n) \sim (-x_0, -x_1, \ldots, -x_n)$.

Lemma 2.3. *The space $\mathbb{RP}^n$ can be thought of as an n-ball where antipodal points on the boundary are identified.*

Proof. Because antipodal points are identified, we can consider the top hemisphere

$$\{(x_0, x_1, \ldots, x_n) \mid x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}/\sim,$$

where $x_n \geq 0$ and $(x_0, x_1, \ldots, x_n) \sim (-x_0, -x_1, \ldots, -x_n)$. In this set, most of the duplicate points have been removed, but in the set

$$\{(x_0, x_1, \ldots, x_{n-1}, 0) \mid x_0^2 + x_1^2 + \ldots x_{n-1}^2 = 1\},$$

antipodal points are still identified. Thus, we can "flatten" this hemisphere into n-dimensional space by projecting it onto the hyperplane defined by $x_n = 0$, and visualize this as an n-dimensional ball where antipodal points are identified. Explicitly, this is the set

$$\{(x_0, x_1, \ldots, x_{n-1}) \mid x_0^2 + x_1^2 + \cdots + x_{n-1}^2 \leq 1\},$$

where $(x_0, x_1, \ldots, x_{n-1}) \sim (-x_0, -x_1, \ldots, -x_{n-1})$. ∎

We are also interested in quandles, which are defined in Definition 3.1, in $\mathbb{RP}^2 \times (0, 1)$. Similarly to $\mathbb{RP}^3$, the manifold $\mathbb{RP}^2$, can be thought of $\mathbb{S}^2/\{\pm 1\}$, or the 2-sphere with antipodal points glued to each other. Thus, we can describe $\mathbb{RP}^2 \times (0, 1)$ using the set

$$\{(x, y, z, t) \mid x^2 + y^2 + z^2 = 1,\ 0 \leq t \leq 1\}/\sim,$$

where $(x, y, z, t) \sim (-x, -y, -z, t)$. Analogously to $\mathbb{RP}^3$, there is also an interpretation of $\mathbb{RP}^2$ as a two-dimensional disk where antipodal points are identified.

2.2 Slide Moves

Similarly to how knots in three-dimensional Euclidean space can be projected to $\mathbb{R}^2$, knots in $\mathbb{RP}^3$ and $\mathbb{RP}^2 \times (0, 1)$ can also be visualized with a two-dimensional diagram.

Definition 2.4. A *tangle* is an embedding of a disjoint union of n arcs and m circles into a 3-ball, where all $2n$ endpoints of the arcs are sent to points on the ball's boundary.

Since $\mathbb{RP}^3$ can be thought of as a three-dimensional ball where antipodal points are glued together, we can project knots from $\mathbb{RP}^3$ into two-dimensional space by expressing them diagrammatically as a tangle where antipodal points are glued, and over-crossings and under-crossings are distinguished from one another.

In $\mathbb{R}^3$, any ambient isotopy of a knot can be expressed in $\mathbb{R}^2$ through a combination of Reidemeister moves and planar isotopies [2]. There is also an analogue of this in $\mathbb{RP}^3$, which we now establish.

Theorem 2.5 ([5]). *In addition to the three Reidemeister moves that describe ambient isotopies of the knot, there are two additional slide moves (described in Figure 2) that define ambient isotopies in* $\mathbb{RP}^3$.

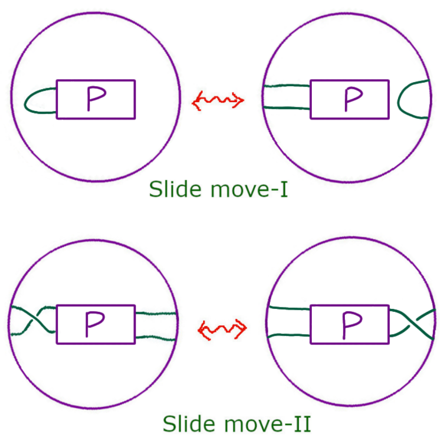

Figure 2: Slide move I pushes a loop through the boundary, and slide move II pushes a crossing through the boundary. Graphic created by the student researcher based on [5, 6] using Ibis Paint X, 2025.

Throughout this paper, the rectangle with a P will represent the rest of the knot that is unaffected by these moves. The P stays upright in Figure 2, showing that the orientation of the rest of the knot does not change.

Theorem 2.6. *In* $\mathbb{RP}^2 \times (0,1)$*, there are also two additional slide moves that help describe ambient isotopies of the knot. Slide move I is identical to that of* $\mathbb{RP}^3$*, and for slide move II in* $\mathbb{RP}^2 \times (0,1)$*, the crossing is oriented differently when it passes through the boundary, as shown below in Figure 3.*

Remark. The reason for the flipped orientation is because in $\mathbb{RP}^3$, antipodal points are identified with each other, so

$$(x, y, z) \sim (-x, -y, -z).$$

On the other hand, in $\mathbb{RP}^2 \times (0,1)$, points (x, y, z) on the boundary get equated to $(-x, -y, z)$ since the coordinate corresponding to the element of $(0,1)$ does not change. Due to this distinction, we must treat the coloring of knots in the manifolds $\mathbb{RP}^3$ and $\mathbb{RP}^2 \times (0,1)$ differently.

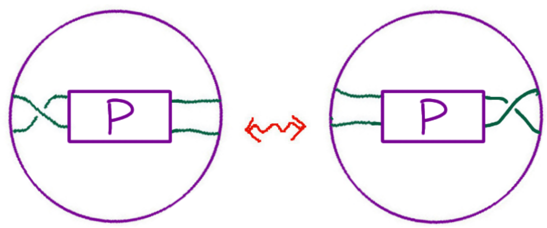

Figure 3: Slide move II in $\mathbb{RP}^2 \times (0,1)$ results in a differently oriented crossing. Graphic created by the student researcher using Ibis Paint X, 2025.

3 Quandles

We are interested in determining whether or not two knots are isotopic, or equivalent, by computing invariants of the knot. One such way is through coloring the knot under a set of rules such that the number of colorings of the knot will stay the same under all three of the Reidemeister moves. Thus, the number of colorings of the knot would be an oriented link invariant. Formally, we will describe such a system with a quandle, where the arcs are colored by elements of a set Q. We now present fundamental properties of quandles.

Definition 3.1 ([1])**.** A *quandle* is a pair $(Q, *)$, where Q is a set and $*\colon Q \times Q \to Q$ is a binary operation on Q satisfying the following conditions:

1. The operation $*$ is idempotent. In other words, for any element $a \in Q$, $a * a = a$.

2. There exists an inverse operation $\overline{*} : Q \times Q \to Q$ such that any elements $a, b \in Q$ satisfy the relations $(a * b) \overline{*} b = a$ and $(b \overline{*} a) * a = b$.

3. The operation $*$ is self-distributive. Given elements $a, b, c \in Q$, then it is true that $(a * b) * c = (a * c) * (b * c)$.

Note that conditions 1, 2, and 3 correspond to the first, second, and third Reidemeister moves, respectively.

Definition 3.2. In an oriented knot diagram D, let $\text{arcs}(D)$ denote the set of arcs of D. Let $\psi\colon \text{arcs}(D) \to Q$ denote a coloring of arcs of D satisfying the conditions in Figure 4.

Definition 3.3. Let $\text{col}_Q(D)$ denote the set of possible ψ for a knot diagram D.

Theorem 3.4 ([1])**.** *The size of* $\text{col}_Q(D)$ *is invariant under each of the Reidemeister moves and therefore under any ambient isotopy of the knot.*

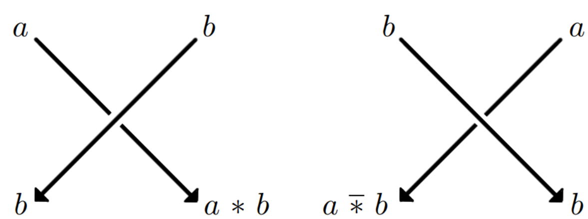

Figure 4: As shown on the left, in a crossing with positive orientation, the color of the third arc is given by $a * b$, and as shown on the right, in a crossing with negative orientation, the color of the third arc is given by $a \overline{*} b$. Graphic created by the student researcher using Ibis Paint X, 2025.

Example 3.5. We can attach the *Joyce quandle* to a group G, with conjugation as the operation. Specifically, let $a * b = bab^{-1}$ and $a \overline{*} b = b^{-1}ab$, as described in [7]. We can verify this satisfies the three axioms of a quandle:

1. The operation is idempotent, because $a * a = a \cdot a \cdot a^{-1} = a$.

2. The inverse operation is valid, because

$$(a * b) \overline{*} b = bab^{-1} \overline{*} b = b^{-1}bab^{-1}b,$$

which is equal to a. It can be similarly shown that $(b \overline{*} a) * a = b$.

3. The quandle operation also satisfies self-distributivity, because

$$\begin{aligned}(a * c) * (b * c) &= (cac^{-1}) * (cbc^{-1}) \\ &= cbc^{-1}cac^{-1}cb^{-1}c^{-1} \\ &= cbab^{-1}c^{-1} \\ &= bab^{-1} * c \\ &= (a * b) * c.\end{aligned}$$

3.1 Fox n-coloring and the Fox Group

Definition 3.6. A *Fox coloring* is a quandle on some abelian group A where we define $a * b = a \mathbin{\overline{*}} b = 2b - a$. If A is of the form $\mathbb{Z}/n\mathbb{Z}$, we call this a *Fox n-coloring.*

In the Fox n-coloring, the condition is commonly expressed as $2b = a + c \pmod{n}$, where a and c are the colors of the undercrossing strands and b is the color of the overcrossing strand, and is illustrated in Figure 1. We can easily verify that the Fox n-coloring satisfies the axioms of a quandle.

Lemma 3.7. *The set of Fox colorings of a knot diagram D by an abelian group A form a group. Given two Fox colorings of the same knot diagram $\psi_1, \psi_2 \in \mathrm{col}_A(D)$, we define $\psi_1 + \psi_2$ to be the coloring such that for each strand $s \in D$, $(\psi_1 + \psi_2)(s) := \psi_1(s) + \psi_2(s)$.*

Moreover, we can verify using the three Reidemeister moves that not only is the *size* of the group a knot invariant, but the *group* itself is also a knot invariant, up to isomorphism. For example, if this group was isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ for one link and isomorphic to $\mathbb{Z}/9\mathbb{Z}$ for another link, the two links can still be distinguished.

Proposition 3.8. *There exists a quandle isomorphism between the Joyce quandle on the reflections in the dihedral group $D_n = \langle r, s \mid r^n = 1, s^2 = 1, srs = r^{-1} \rangle$ and the Fox n-coloring quandle $F_{\mathbb{Z}/n\mathbb{Z}}$.*

Proof. Let Q_{D_n} be colored with the set $\{s, sr, \ldots, sr^{n-1}\}$. Then if we let two elements $a, b \in Q_{D_n}$ be expressed as $a = sr^k$ and $b = sr^\ell$, we can compute that

$$a * b = bab^{-1} = sr^\ell \cdot sr^k \cdot sr^\ell = sr^{2\ell - k},$$

suggesting that there is a quandle isomorphism $\varphi\colon Q_{D_n} \to F_{\mathbb{Z}/n\mathbb{Z}}$ where $\varphi(sr^k) = k$. ■

More generally, there is also a notion of a fundamental group of the Fox coloring.

Definition 3.9 ([1])**.** Let $\mathcal{F}_D$ be the *fundamental group of the Fox coloring* of an oriented knot diagram D. Define it to be the group generated by arcs(D), the set of arcs of D, with the set of relations such that at every crossing with under-strands x_i and x_k and over-strand x_j, we have $x_i + x_k = 2x_j$.

Proposition 3.10. *The fundamental group of the Fox coloring is a knot invariant. In particular, the group of Fox colorings of a link diagram by an abelian group A is isomorphic to the group of homomorphisms from $\mathcal{F}_D$ to A.*

We can verify this using the three Reidemeister moves—due to the structure of the quandle, each of the moves will not add additional information to the set of relations of the link diagram. Thus, the fundamental group of Fox colorings is a more general way to understand the Fox colorings of a knot.

3.2 Quandles in $\mathbb{RP}^2 \times (0, 1)$

We now extend the notion of a quandle to the manifold $\mathbb{RP}^2 \times (0, 1)$, and analogously the notions of the Fox coloring and the Joyce quandle. We call two strands *antipodal* if they are connected to antipodal points on the boundary of the disk.

Definition 3.11. Let Q be a quandle and let $\varphi\colon Q \to Q$ be an involution satisfying the property that $\varphi(a*b) = \varphi(a) \mathbin{\bar{*}} \varphi(b)$. Then a (Q, φ)*-coloring* is a coloring of the knot diagram such that if a strand passes through a boundary point and is colored with c, then the strand passing through the antipodal point is colored with $\varphi(c)$.

We want φ to be an involution because we are given that if a strand x is colored with c, the antipodal strand x' is colored by $\varphi(c)$. Then the strand antipodal to x' must be colored by $\varphi(\varphi(c))$. However, the strand antipodal to x' is just x, which we know is colored with c. Thus, we obtain the condition that $\varphi(\varphi(c)) = c$.

Theorem 3.12. *For any quandle Q and involution φ such that $\varphi(a * b) = \varphi(a) \mathbin{\bar{*}} \varphi(b)$, the number of (Q, φ)-colorings of a link in $\mathbb{RP}^2 \times (0, 1)$ is an oriented link invariant.*

Proof. We want to show that the number of colorings under (Q, φ) is invariant under both slide moves. In particular, as shown in Figure 5, the condition corresponding to slide move I is that $\varphi(a) = \varphi(b)$ implies $a = b$, or simply that φ is injective. Since φ is an involution, this condition is automatically satisfied.

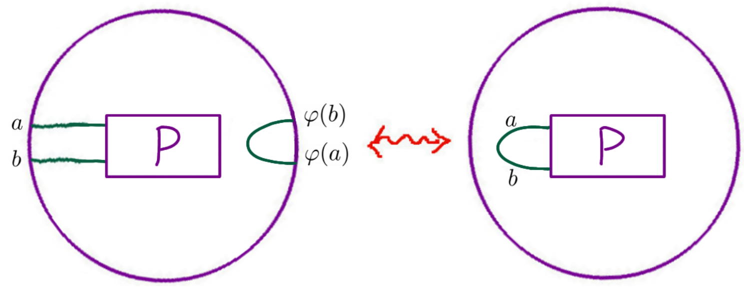

Figure 5: Condition arising from slide move I in $\mathbb{RP}^2 \times (0,1)$. Graphic created by the student researcher using Ibis Paint X, 2025.

Figure 6 gives us a direct bijection between colorings of the knot before and after slide move II is performed, as long as $\varphi(a * b) = \varphi(a) \,\overline{*}\, \varphi(b)$. Thus, because there are explicit

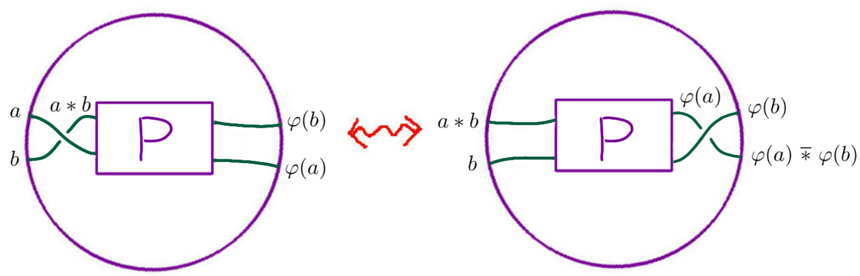

Figure 6: Condition arising from slide move II in $\mathbb{RP}^2 \times (0,1)$. Graphic created by the student researcher using Ibis Paint X, 2025.

bijections for colorings under the slide moves, the number of colorings is an oriented link invariant. ■

Corollary 3.13. *Let K be a knot in $\mathbb{RP}^2 \times (0,1)$. If K is colored under the Joyce quandle, with the additional constraint that the colors of arcs connected to antipodal points must be inverses of each other, the number of colorings forms a knot invariant.*

Proof. Consider the Joyce quandle, described in Example 3.5. We can verify that this satisfies the three axioms of a quandle, and we can extend this to knots in $\mathbb{RP}^2 \times (0,1)$ by defining $\varphi(a) = a^{-1}$. Then we can see that

$$\varphi(a) \,\overline{*}\, \varphi(b) = ba^{-1}b^{-1} = (bab^{-1})^{-1} = \varphi(a * b).$$

Thus, φ is an involution that satisfies the relation demonstrated in Figure 6. ■

Remark. In $\mathbb{RP}^3$, a system of coloring similar to the one described in Theorem 3.12 force Q to be the trivial quandle. Consider the condition that arises from slide move II in $\mathbb{RP}^3$, as shown in Figure 7.

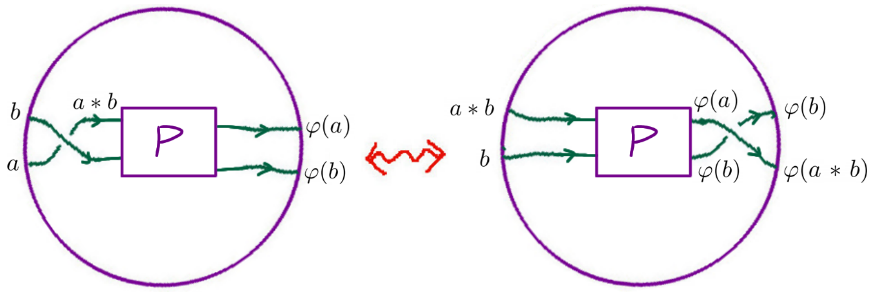

Figure 7: In $\mathbb{RP}^3$, slide move II tells us that the quandle must be trivial. Graphic created by the student researcher using Ibis Paint X, 2025.

Notice that if we want the number of colorings in both representations of the knot to be the same, it must be true that $\varphi(b) * \varphi(a) = \varphi(b)$ for all $a, b \in Q$, which means that Q must be the trivial quandle.

We can use this quandle to differentiate between two links in $\mathbb{RP}^2 \times (0, 1)$ that have the same preimage.

Proposition 3.14. *When colored by the Joyce quandle with the set* $\{s, sr, sr^2, sr^3\} \subset D_4$ *(which, as described in Proposition 3.8, is isomorphic to the Fox 4-coloring), the two links shown in Figure 8 (which have the same preimage) can be differentiated.*

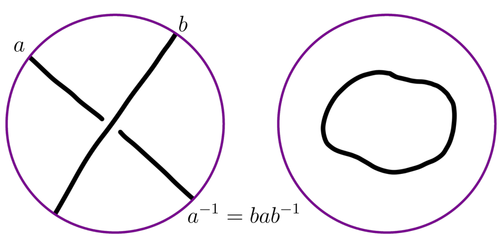

Figure 8: Two different links in $\mathbb{RP}^2 \times (0, 1)$ that have the same preimage. Graphic created by the student researcher using Ibis Paint X, 2025.

Proof. For the link on the left, let $a = sr^k$ and $b = sr^\ell$. Then we have the condition that $a^{-1} = bab^{-1}$. Thus,

$$sr^k = sr^\ell sr^k sr^\ell,$$

which simplifies to $sr^k = sr^\ell r^{-k} r^\ell$. This condition implies that $2k \equiv 2\ell \pmod 4$, so $k \equiv \ell \pmod 2$. We can then see that there are 8 possible colorings, since there are 4 possible values of k and 2 possible values of ℓ for each value of k. However, the affine unknot in $\mathbb{RP}^2 \times (0,1)$ (on the right) has 4 possible colorings, so both links must be different. ■

3.3 Fox n-colorings in $\mathbb{RP}^2 \times (0,1)$

Analogously to the fundamental group of Fox colorings in $\mathbb{S}^3$, we can also define a fundamental $\mathbb{Z}[t]/(t^2-1)$-module of Fox colorings in $\mathbb{RP}^2 \times (0,1)$. Recall that a $\mathbb{Z}[t]$ module on an abelian group A is defined with an endomorphism $\varphi(x) = tx$, where $x, tx \in A$. In a $\mathbb{Z}[t]/(t^2-1)$-module, we have the additional constraint that $\varphi^2(x) = 1$, so φ must be an involutive automorphism.

Lemma 3.15. *There is a categorical equivalence between the category of $\mathbb{Z}[t]/(t^2-1)$-modules and the category of a pair (A, φ) of an abelian group A and an involutive automorphism φ.*

Definition 3.16. The *fundamental $\mathbb{Z}[t]/(t^2-1)$-module of Fox colorings $\mathscr{F}_D$* of a knot diagram D in $\mathbb{RP}^2 \times (0,1)$ is the $\mathbb{Z}[t]/(t^2-1)$-module generated by arcs(D) and governed by the following relations:

- At every crossing, if x_i and x_k are the colors of the undercrossing strands and x_j is the color of the overcrossing strand, then $x_i + x_k = 2x_j$.
- If x and x' are antipodal strands, then $x' = \varphi(x)$, where φ is the corresponding involutive automorphism.

Similar to the fundamental group of Fox colorings in $\mathbb{S}^3$, the fundamental $\mathbb{Z}[t]/(t^2-1)$-module of Fox colorings in $\mathbb{RP}^2 \times (0,1)$ helps us better understand the set of Fox colorings of a knot diagram in $\mathbb{RP}^2 \times (0,1)$. Analogously, every such Fox n-coloring can be thought of as a homomorphism from $\mathscr{F}_D$ to the $\mathbb{Z}[t]/(t^2-1)$ module of A.

4 Dehn colorings

Recall the Dehn presentation of a knot, as described in Definition 4.1. The link invariants described in this sections are colorings of regions motivated by the Dehn presentation of the fundamental group of the knot complement.

4.1 Presentation for the Fundamental Group of the Link Complement in $\mathbb{RP}^3$

The fundamental group of the complement of a link L in a manifold M, denoted $\pi_1(M \setminus L)$, is an important object in knot theory, especially because it is close to being a complete link invariant [8]. Thus, our knot invariants may be motivated by homomorphisms from $\pi_1(M \setminus L)$ to groups with simpler structures that are easier to understand, such as the cyclic group or the dihedral group.

One presentation for a link complement in $\mathbb{S}^3$ is Dehn's presentation, which is defined as follows.

Definition 4.1 (Dehn's presentation [1, 9])**.** Consider a checkerboard coloring of regions bounded by arcs of the link diagram of a link L. Then $\pi_1(\mathbb{S}^3 \setminus L)$ is the group of loops starting from a base point P in the outer region, up to homotopy. The generators of the group correspond to the paths starting from P and going through the regions such that the path is *positively oriented* if the region is *light* and *negatively oriented* if the region is *dark*.

Additionally, we have a relation for each crossing that begins from a dark region, then travels to the adjacent light region separated by the overcrossing arc, then goes to the next adjacent dark region and light region.

For example, in Figure 9 we get the relation that $cdab = 1$, where a, b, c, and d are generators corresponding to the regions they pass through. We could also equivalently get the relation that $abcd = 1$.

We are interested in a presentation for the complement of a link in $\mathbb{RP}^3$. This presentation is a generalization of Dehn's presentation for the link complement in $\mathbb{S}^3$.

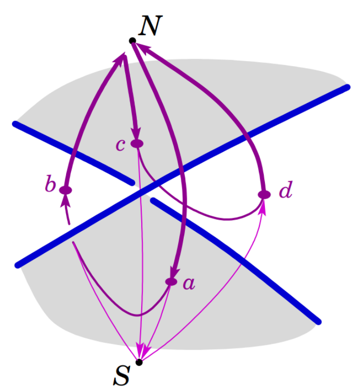

Figure 9: Relation $abcd = 1$ at a crossing in a Dehn presentation of a link. Graphic created by the student researcher based on [9] using Ibis Paint X, 2025.

Definition 4.2 (Presentation of $\pi_1(\mathbb{RP}^3 \setminus L)$ [9])**.** Consider the group of loops starting from a base point P (up to homotopy), which we choose to be the point defined by the north pole N and south pole S of the ball corresponding to $\mathbb{RP}^3$. Choose a checkerboard coloring of regions of the projection of the link. Similarly to in $\mathbb{S}^3$, the generators of the group are as follows:

- If the region is dark, the generator is the path through the region from N to S.
- If the region is light, the generator is the path through the region from S to N.

These generate $\pi_1(\mathbb{RP}^3 \setminus L)$, and we have additional relations based on the crossings and antipodal regions, as follows:

- There is the same relation at each crossing as in the Dehn presentation of the link complement in $\mathbb{S}^3$, shown in Figure 9.
- If a and b are generators corresponding to regions of the same color adjacent to antipodal arcs, then $a = b^{-1}$.
- If a and b are generators corresponding to regions of different colors adjacent to antipodal arcs, then $a = b$.

4.2 Dehn colorings in $\mathbb{S}^3$

Given a link L, consider a homomorphism from $\pi_1(\mathbb{S}^3 \setminus L)$ to the dihedral group D_n such that generators corresponding to dark regions map to elements of D_n are of the form $r^i s$ and generators corresponding to light regions correspond to elements of the form r^j, as shown in Figure 10.

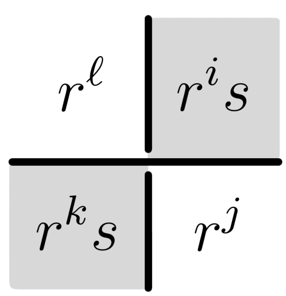

Figure 10: One possible mapping from $\pi_1(\mathbb{S}^3 \setminus L)$ to D_n. The shaded and unshaded regions correspond to the dark and light regions, respectively, in the checkerboard coloring of the knot. Graphic created by the student researcher using Ibis Paint X, 2025.

Then the condition we get from the relation corresponding to this crossing is $r^i s r^j r^k s r^\ell = 1$. Because the dihedral group has the relation that $srs = r^{-1}$, we can simplify this condition to

$$r^i s r^j r^k s r^\ell = r^i s r^{j+k} s r^\ell = r^i r^{-j-k} r^\ell = r^{i-j-k+\ell} = 1,$$

which tells us that $i - j - k + \ell = 0$. Another possible convention is to map dark regions to elements of the form r^i and light regions to elements of the form $r^j s$, which would tell us that $i + j - k - \ell = 0$.

This motivates the following link invariant:

Definition 4.3 (Dehn coloring)**.** Let A be any abelian group. Define a Dehn coloring to be a coloring of the regions of the link projection such that at any crossing, the sum of the colors on one side of the overcrossing is equal to the sum of the colors on the other side of the overcrossing. Specifically, in Figure 11, we get the condition that $c_1 + c_2 = c_3 + c_4$ at every crossing.

Theorem 4.4. *The number of Dehn colorings is a link invariant.*

This can be shown by considering how the colorings change under the three Reidemeister moves.

4.3 Dehn colorings in $\mathbb{RP}^3$

Similarly to the Dehn colorings in $\mathbb{S}^3$, we can extend this idea to $\mathbb{RP}^3$ by coloring the regions formed by the link projection in a way that is based on the Dehn presentation of the fundamental group of the link complement.

Lemma 4.5. *The number of boundary points in a diagram of a link in $\mathbb{RP}^3$ mod 4 is invariant.*

Proof. Out of the three Reidemeister moves and the two slide moves, slide move I is the only one that changes the number of boundary points. Since slide move I changes the number of boundary points by 4, this implies the result. ■

Theorem 4.6. *Let L be a link such that the number of boundary points on a projection of L is divisible by 4. Color the regions of the link projection by elements of an abelian group A, and let φ be an involutive automorphism on A. At each crossing, suppose the regions on one side of the overcrossing are colored by c_1 and c_2 and the regions on the other side are colored by c_3 and c_4, as shown in Figure 11. Then the number of colorings of the regions by elements of A conforming to the following conditions is a link invariant in $\mathbb{RP}^3$.*

- *At every crossing,* $c_1 + c_2 + \varphi(c_3) + \varphi(c_4) = 0$.
- *If* a *and* a' *are the colors of regions adjacent to antipodal arcs:*
  - *If* a *and* a' *are both dark, then* $a' = a$.
  - *If* a *and* a' *are both light, then* $a' = \varphi(a)$.

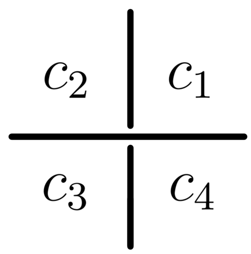

Figure 11: Colorings of regions around a crossing. Graphic created by the student researcher using Ibis Paint X, 2025.

Proof. By Lemma 4.5, the number of boundary points is divisible by 4. Thus, regions connected to each other through antipodal arcs will always be the same color under a checkerboard coloring of the knot diagram. Therefore, the condition is well-defined.

The proof that the number of colorings is invariant under the three Reidemeister moves is the analogous to the proof for the classical case in $\mathbb{S}^3$ where $\varphi(c) = -c$; since the Reidemeister moves do not affect the boundary points, these are unaffected by the conditions pertaining to regions adjacent to antipodal arcs.

Slide move I gives us a direct bijection between colorings of the knot diagrams. If the region inside the loop is dark, then if it is colored with a, it must still be colored with a when pulled through the boundary, as shown in Figure 12.

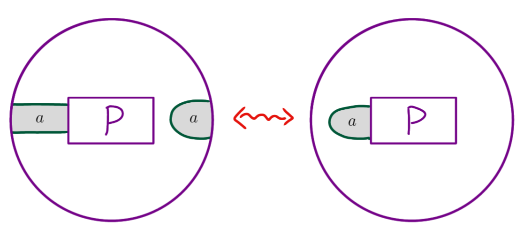

Figure 12: Bijection between colorings under slide move I when the region is dark. Graphic created by the student researcher using Ibis Paint X, 2025.

Similarly, if the region inside the loop is light and is colored with a, the region adjacent to the antipodal arc when the loop is pulled through the boundary must be colored with $\varphi(a)$, as shown in Figure 13.

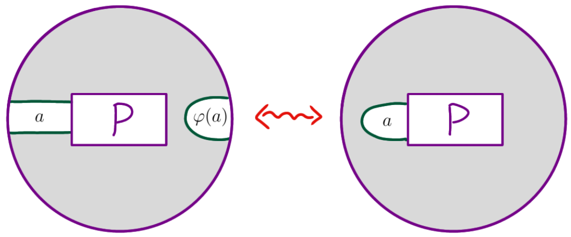

Figure 13: Bijection between colorings under slide move I when the region is light. Graphic created by the student researcher using Ibis Paint X, 2025.

Slide move II also preserves the number of Dehn colorings of the link diagram, as shown in Figure 14.

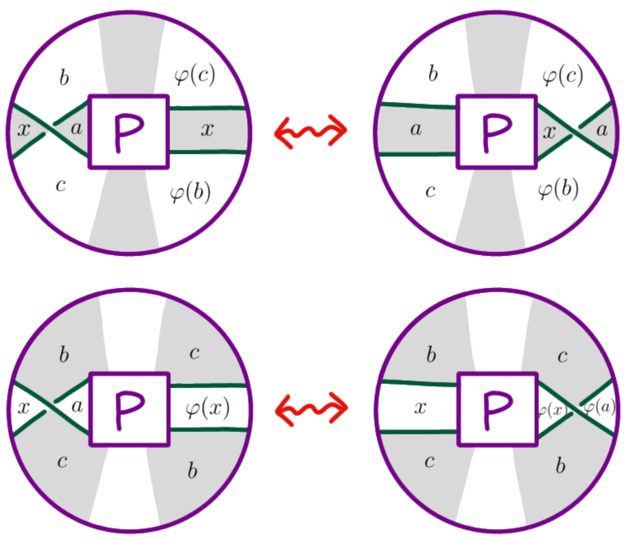

Figure 14: Bijection between colorings under slide move II. Graphic created by the student researcher using Ibis Paint X, 2025.

Suppose that at a crossing next to the boundary, the colors of the regions around it satisfy $a + b + \varphi(x) + \varphi(c) = 0$, as shown on the left of Figure 14. When the regions adjacent to the boundary arcs are dark, after slide move II is performed, the result is shown on the top of Figure 14, and it gives us the condition that

$$a + \varphi(c) + \varphi(x) + \varphi(\varphi(b)) = a + \varphi(c) + \varphi(x) + b = 0,$$

which is equivalent to the original condition. Similarly, when the regions adjacent to the boundary arcs are light, as shown on the bottom left of Figure 14, we get the condition

$$b + \varphi(x) + \varphi(c) + \varphi(\varphi(a)) = b + \varphi(x) + \varphi(c) + a = 0,$$

after the move is performed, which is also equivalent to the original condition. Thus, we get a bijection between colorings of the link before and after the move for either checkerboard pattern of the diagram.

Since the total number of colorings will stay the same under all five possible moves, we have a link invariant. ■

Remark. The condition on colors of antipodal regions is motivated by a homomorphism from $\pi_1(\mathbb{RP}^3 \setminus L)$ to $D_n = \mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$ similar to the one described in Section 4.2. Because of how the presentation of the fundamental group of the link complement is defined in $\mathbb{RP}^3$, in addition to the condition for Dehn colorings in $\mathbb{S}^3$, we also want the generators corresponding to antipodal regions in D_n to be inverses of each other, since they are assumed to be the same color and this corresponds to the condition stated in Definition 4.2. Thus, since dark regions are colored by reflections (which are their own inverse), we have $a' = a$ for dark antipodal regions, and since light regions are colored by rotations (where r^{-k} is the inverse of r^k), we have the generalized condition $a' = \varphi(a)$ for light antipodal regions.

4.4 Dehn colorings in $\mathbb{RP}^2 \times (0,1)$

We now pursue analogues of these results in $\mathbb{RP}^2 \times (0,1)$ by coloring regions of link diagrams in $\mathbb{RP}^2 \times (0,1)$.

Theorem 4.7. *Let A be an abelian group, and φ be an involutive automorphism on A. Then the number of Dehn colorings of a link diagram as described in Definition 4.3, with the additional constraint that if a and a' are the colors of regions adjacent to antipodal arcs, then $a' = \varphi(a)$, is a link invariant.*

Notably, unlike in $\mathbb{RP}^3$, we do not require the number of boundary points to be a multiple of 4, nor do we assign a checkerboard coloring to the link diagram.

Proof. Because the Reidemeister moves do not affect the boundary points, the number of colorings should stay the same under any of the three Reidemeister moves since we have already shown this in Theorem 4.4.

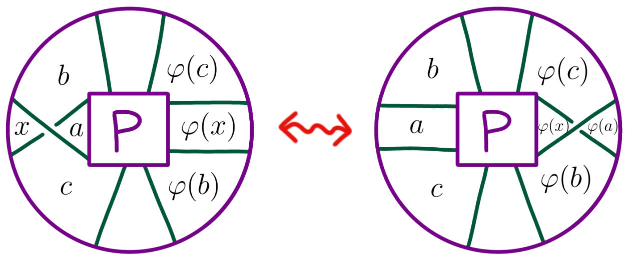

Figure 15: Bijection between colorings under slide move II in $\mathbb{RP}^2 \times (0,1)$. Graphic created by the student researcher using Ibis Paint X, 2025.

The proof that slide move I preserves the number of colorings is the same as the one demonstrated in Figure 13.

To show the slide move II keeps the number of colorings the same, consider Figure 15. On the left, we have the condition that $a + b + \varphi(x) + \varphi(c) = 0$. Lastly, for the second slide move, again let the regions be colored in a way that satisfies the conditions in Theorem 4.6. When the crossing is pulled over the boundary and the rest of the coloring is not disturbed, the condition is still satisfied since

$$\varphi(x) + \varphi(c) + \varphi(\varphi(a)) + \varphi(\varphi(b)) = \varphi(x) + \varphi(c) + a + b = 0.$$

Thus, there is a bijection between colorings before and after slide move II, so the number of colorings must be invariant. ■

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