EXCEPTIONAL ISOMORPHISMS BETWEEN FINITE GROUPS

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Abstract. I will explicitly construct exceptional isomorphisms between finite groups. I will present the most "natural" constructions I know.

I will only construct the homomorphisms since checking they are isomorphisms is straightforward.

1. Warm-up

1.1. $S_3 \simeq \mathrm{GL}_2(\mathbb{F}_2)$**.** The group $\mathrm{GL}_2(\mathbb{F}_2)$ acts on the 3-element set $\mathbb{P}^1(\mathbb{F}_2)$ which produces a homomorphism $\mathrm{GL}_2(\mathbb{F}_2) \to S_3$. Conversely, S_3 always has the standard 2-dimensional representation: it acts on the 2-dimensional $\mathbb{F}_2$-vector space $\{(x, y, z) \in \mathbb{F}_2^3 : x + y + z = 0\}$ by permuting the entries.

1.2. $S_4 \simeq \mathrm{PGL}_2(\mathbb{F}_3)$**.** Again the group $\mathrm{PGL}_2(\mathbb{F}_3)$ acts on the 4-element set $\mathbb{P}^1(\mathbb{F}_3)$ which produces a homomorphism $\mathrm{PGL}_2(\mathbb{F}_3) \to S_4$. Conversely, S_4 acts on the standard 3-dimensional representation $\{(x_1, \ldots, x_4) \in \mathbb{F}_3^4 : \sum_{i=1}^4 x_i = 0\}$ by permuting coordinates. This vector space carries an invariant orthogonal form $(x_1, \ldots, x_4) \cdot (y_1, \ldots, y_4) = \sum_{i=1}^4 x_i y_i$, so this defines a homomorphism $S_4 \to \mathrm{SO}_3(\mathbb{F}_3)$. Then we use the exceptional isomorphism of algebraic groups $\mathrm{PGL}_2 \simeq \mathrm{SO}_3$. Here we need to use that the orthogonal form is isotropic: the vector $(1, 1, 1, 0)$ has dot product 0 with itself.

1.3. $S_4 \simeq \mathbb{F}_2^2 \rtimes \mathrm{GL}_2(\mathbb{F}_2)$**.** Note that $\mathbb{F}_2^2 \rtimes \mathrm{GL}_2(\mathbb{F}_2)$ is the group of affine linear transformations on $\mathbb{F}_2^2$. Since $\mathbb{F}_2^2$ has size 4 this produces a homomorphism $\mathbb{F}_2^2 \rtimes \mathrm{GL}_2(\mathbb{F}_2) \to S_4$.

Remark 1.4. Recall that there is an exceptional surjection $S_4 \to S_3$ since there are 3 ways to partition a four-element set into two equal subsets. Under the exceptional isomorphisms 1.1 and 1.3 this surjection corresponds onto the projection to the second factor of $\mathbb{F}_2^2 \rtimes \mathrm{GL}_2(\mathbb{F}_2)$.

1.5. $A_5 \simeq \mathrm{PSL}_2(\mathbb{F}_4)$**.** Again the group $\mathrm{PSL}_2(\mathbb{F}_4)$ acts on the 5-element set $\mathbb{P}^1(\mathbb{F}_4)$ which produces a homomorphism $\mathrm{PSL}_2(\mathbb{F}_4) \to A_5$.

2. Larger groups

2.1. $S_5 \simeq \mathrm{PGL}_2(\mathbb{F}_5)$**.** Constructing the homomorphism is harder, since $\mathrm{PGL}_2(\mathbb{F}_5)$ naturally acts on the 6-element set $\mathbb{P}^1(\mathbb{F}_5)$. Our goal will be to construct an action of $\mathrm{PGL}_2(\mathbb{F}_5)$ on a 5-element set.

Remark 2.2. In fact, Galois, in his letter to Chevallier written the evening before his fatal dual, observes that $\mathrm{PSL}_2(\mathbb{F}_p)$ always acts on $p + 1$ points but only acts on p points when $p = 5, 7, 11$.

First, observe that there are $15 = 6!/(2^3 \cdot 3!)$ fixed-point-free involutions on the 6-element set $\mathbb{P}^1(\mathbb{F}_5)$. Of those, 10 of them arise from an element of $\mathrm{PGL}_2(\mathbb{F}_5)$. Indeed, they are given by matrices in $\mathrm{PGL}_2(\mathbb{F}_5)$ which are of order 2 and do not have any eigenvectors. That means g must be conjugate to $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, the image of the order 2 element in $\mathbb{F}_{25}^\times/\mathbb{F}_5^\times \hookrightarrow \mathrm{PGL}_2(\mathbb{F}_5)$. The centralizer of such a matrix is $\mathbb{F}_{25}^\times/\mathbb{F}_5^\times \rtimes \mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$ so its orbit is identified with $\mathrm{GL}_2(\mathbb{F}_5)/(\mathbb{F}_{25}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5))$ which has size 10.

Now there are 5 fixed-point-free-involutions on $\mathbb{P}^1(\mathbb{F}_5)$ not induced from $\mathrm{PGL}_2(\mathbb{F}_5)$. These are permuted by conjugation by $\mathrm{PGL}_2(\mathbb{F}_5)$ and produces a homomorphism $\mathrm{PGL}_2(\mathbb{F}_5) \to S_5$.

Remark 2.3. Since $\mathrm{PGL}_2(\mathbb{F}_5)$ acts on 6 points naturally, the above constructs an exotic embedding $S_5 \simeq \mathrm{PGL}_2(\mathbb{F}_5) \hookrightarrow S_6$. The action of S_6 on the quotient $S_6/\mathrm{PGL}_2(\mathbb{F}_5)$ produces the exceptional outer automorphism of S_6! In fact all automorphisms of S_n are inner except for when $n = 6$, when $\mathrm{Out}(S_6)$ has order two.

2.4. $A_6 \simeq \mathrm{PSL}_2(\mathbb{F}_9)$**.** A group-theoretic construction is that $\mathrm{PSL}_2(\mathbb{F}_9)$ has 6 Sylow 5-subgroups, and the conjugation action produces a homomorphism $\mathrm{PSL}_2(\mathbb{F}_9) \to A_6$. These Sylow 5-subgroups arise from the observation that the norm 1 elements $(\mathbb{F}_{81}^\times)^{\mathrm{Nm}=1}$ in $\mathbb{F}_{81}^\times$ form a group of order 10, so the quotient $(\mathbb{F}_{81}^\times)^{\mathrm{Nm}=1}/\{\pm 1\}$ is a cyclic subgroup of $\mathrm{PSL}_2(\mathbb{F}_9)$ of order 5.

A more geometric construction observes that A_5 acts on the dodecahedron via rotation, which produces a homomorphism $A_5 \to \mathrm{SO}_3(\mathbb{R})$. That the automorphism group of the dodecahedron is A_5 follows from the fact that there are 5 embedded cubes in the dodecahedron. Furthermore, there is a nice set of vertices of the dodecahedron: $(0, \pm 1, \pm\phi)$ and cyclic permutations, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. This means that the representation is actually defined over $\mathbb{Z}[\frac{1}{2}, \phi]$. Let's extend scalars to $\mathbb{Z}[\frac{1}{2}, i, \phi]$ so that the orthogonal form becomes isotropic and reduce mod 3. This produces a homomorphism $A_5 \hookrightarrow \mathrm{PSL}_2(\mathbb{F}_9)$ since $\mathbb{F}_9 = \mathbb{F}_3[\sqrt{5}, \sqrt{-1}]$. This in particular means $\mathrm{PSL}_2(\mathbb{F}_9)$ acts on the 6-element set $\mathrm{PSL}_2(\mathbb{F}_9)/A_5$ producing a homomorphism $\mathrm{PSL}_2(\mathbb{F}_9) \to A_6$.

Remark 2.5. Just as in Remark 2.3, for most values of n there is only one automorphism of A_n arising from conjugation by S_n. However when $n = 6$ the outer automorphism group is $C_2 \times C_2$. This is visible from the isomorphism $A_6 \simeq \mathrm{PSL}_2(\mathbb{F}_9)$ since $\mathrm{PSL}_2(\mathbb{F}_9)$ has outer automorphisms arising from $\mathrm{PGL}_2(\mathbb{F}_9)$ and from the Galois action.

2.6. $\mathrm{PSL}_2(\mathbb{F}_7) \simeq \mathrm{GL}_3(\mathbb{F}_2)$**.** [Elk99] gives a geometric construction through the Klein quartic. The point is that the automorphism group G of the Klein quartic $X^3Y + Y^3Z + Z^3X = 0$ has order 168. This is the unique genus 3 curve achieving the maximum of the Hurwitz bound which says a degree g curve has at most $84(g-1)$ automorphisms. This gives a 3-dimensional representation of G which can be realized over the ring of integers of $\mathbb{Q}(\sqrt{-7})$. Reducing modulo 2 gives an isomorphism $G \simeq \mathrm{GL}_3(\mathbb{F}_2)$. On the other hand the reduction mod 7 preserves an orthogonal form which gives an isomorphism $G \to \mathrm{SO}_3(\mathbb{F}_7)$. Again recall the exceptional isomorphism of algebraic groups $\mathrm{SO}_3 \simeq \mathrm{PGL}_2$ which shows $G \simeq \mathrm{PSL}_2(\mathbb{F}_7)$.

Remark 2.7. Since $\mathrm{GL}_3(\mathbb{F}_2)$ naturally acts on the 7-element set $\mathbb{P}^2(\mathbb{F}_2)$ this produces an exceptional action of $\mathrm{PSL}_2(\mathbb{F}_7)$ on 7 points. This is the $p = 7$ case of Galois's observation, see Remark 2.2.

2.8. $S_8 \simeq \mathrm{O}_6(\mathbb{F}_2)$**.** Let S be a 8-element set. Let V be the quotient of $\{\sum_{i\in S} x_i e_i \in \mathbb{F}_2^S : \sum_{i\in S} x_i = 0\}$ by $\sum_{i\in S} e_i$. Then V is a 6-dimensional vector space with a quadratic form $q(\sum_{i\in S} x_i e_i) = \sum_{\{i,j\}\subset S} x_i x_j$. Since q is an orthogonal form invariant under S_8 this gives a homomorphism $S_8 \to \mathrm{O}_6(\mathbb{F}_2)$.

2.9. $A_8 \simeq \mathrm{GL}_4(\mathbb{F}_2)$**.** Combine the above isomorphism with the exceptional isomorphism of algebraic groups $\mathrm{SL}_4/\mu_2 \simeq \mathrm{SO}_6$.

Remark 2.10. Since $\mathrm{PSL}_2(\mathbb{F}_7)$ acts on the 8-element set $\mathbb{P}^1(\mathbb{F}_7)$ it can be viewed as a subgroup of A_8. Under the exceptional isomorphism $A_8 \simeq \mathrm{GL}_4(\mathbb{F}_2)$ the subgroup is taken to the obvious subgroup $\mathrm{GL}_3(\mathbb{F}_2)$.

References

[Elk99] Noam D. Elkies, *The Klein quartic in number theory*, The eightfold way, Math. Sci. Res. Inst. Publ., vol. 35, Cambridge Univ. Press, Cambridge, 1999, pp. 51–101. MR 1722413

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