THE STACK OF LOCAL SYSTEMS AND IND-COHERENT SHEAVES

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Abstract. The goal of this talk is to formalize the spectral side of the geometric Langlands equivalence. We define the stack $\mathrm{LS}_{\check{G}}$ of de Rham local systems and prove its basic properties. Beilinson and Drinfeld's "best hope" for the geometric Langlands equivalence uses quasi-coherent sheaves on $\mathrm{LS}_{\check{G}}$, but that category is too small for the equivalence. We therefore enlarge $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ to the category of ind-coherent sheaves on $\mathrm{LS}_{\check{G}}$ with nilpotent singular support.

Let X be a smooth projective curve over a field k of characteristic zero, let G be a split reductive group over k, and let $\check{G}$ be its Langlands dual. Beilinson and Drinfeld's "best hope" for the geometric Langlands equivalence states

$$\text{D-mod}_{\frac{1}{2}}\big(\mathrm{Bun}_G(X)\big) \simeq \mathrm{QCoh}\big(\mathrm{LS}_{\check{G}}(X)\big),$$

It turns out this equivalence cannot hold since the right-hand side is too small. Our goal is to make the right-hand side more precise. Namely, we:

(1) define the stack $\mathrm{LS}_{\check{G}}(X)$ of de Rham $\check{G}$-local systems on X; and
(2) modify the category $\mathrm{QCoh}\big(\mathrm{LS}_{\check{G}}(X)\big)$ to $\mathrm{IndCoh}_{\mathrm{Nilp}}\big(\mathrm{LS}_{\check{G}}(X)\big)$, the category of *ind-coherent sheaves with nilpotent singular support.*

1. the de Rham stack

We want the stack $\mathrm{LS}_{\check{G}}(X)$ to classify $\check{G}$-bundles on X with a flat connection. How can we formalize this?

One clean way is to use Simpson's *de Rham stack* [Sim96]. Let $\mathcal{Y}$ be a smooth scheme over a field k of characteristic zero. We consider its functor of points

$$\mathrm{Rings} \to \mathrm{Sets} : R \mapsto \mathcal{Y}(R)$$

and let $\mathcal{Y}_{\mathrm{dR}}$ be the stack

$$\mathrm{Rings} \to \mathrm{Sets} : R \mapsto \mathcal{Y}(R_{\mathrm{red}}), \tag{1.1}$$

obtained by "identifying infinitesimally close points of $\mathcal{Y}$ together." Then Simpson observed that de Rham cohomology is the coherent cohomology of $\mathcal{Y}_{\mathrm{dR}}$

$$H^i_{\mathrm{dR}}(\mathcal{Y}) \simeq H^i(\mathcal{Y}_{\mathrm{dR}}, \mathcal{O}_{\mathcal{Y}_{\mathrm{dR}}}),$$

and in fact there is an equivalence between the categories of coefficients

$$\text{D-mod}(\mathcal{Y}) \simeq \mathrm{QCoh}(\mathcal{Y}_{\mathrm{dR}}).$$

Now, de Rham local systems on $\mathcal{Y}$ are vector bundles on $\mathcal{Y}_{\mathrm{dR}}$, and $\check{G}$-local systems on $\mathcal{Y}$ are $\check{G}$-bundles on $\mathcal{Y}_{\mathrm{dR}}$, which are maps $\mathcal{Y}_{\mathrm{dR}} \to \mathbb{B}\check{G}$. We thus arrive at the following definition.

Definition 1.1 ([AG15, §10.1.1])**.** Let $\mathcal{Y}$ be a smooth scheme over a field k of characteristic zero, and let $\check{G}$ be a linear algebraic group over k. Let $\mathcal{Y}_{\mathrm{dR}}$ be the de Rham stack, as defined in (1.1). Then the *stack of $\check{G}$-local systems* is defined by $\mathrm{LS}_{\check{G}}(\mathcal{Y}) := \mathrm{Map}(\mathcal{Y}_{\mathrm{dR}}, \mathbb{B}\check{G})$, for any (DG-)scheme S,

$$\mathrm{LS}_{\check{G}}(\mathcal{Y})(S) := \mathrm{Map}(\mathcal{Y}_{\mathrm{dR}} \times S, \mathbb{B}\check{G}).$$

This is a functor $(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Ani}$, where $\mathrm{DGSch}^{\mathrm{aff}}$ is the category of affine DG-schemes and Ani is the ∞-category of anima (or spaces, or ∞-groupoids).

Example 1.2. We really need to work with DG schemes, since $\mathrm{LS}_{\check{G}}$ is a derived object. For example when $X = \mathbb{P}^1$ and $\check{G} = \mathbb{G}_m$, let us consider what connections exist on the line bundle $\mathcal{O}$. Such a connection is of the form $d + f(z)dz$, where $f(z) = \sum_{i\in\mathbb{Z}} a_i z^i$. Since f must be regular, we need

$$a_{-1} = a_{-2} = a_{-3} = \cdots = 0.$$

On the other hand, if we change variables, writing $z = w^{-1}$, we see $d + f(z)dz = d - f(w^{-1})w^{-2}dw$. So for $f(w^{-1})w^{-2}$ to be regular, we need

$$a_{-1} = a_0 = a_1 = \cdots = 0.$$

Since we required $a_{-1} = 0$ *twice*, we have a derived factor of $\Omega\mathbb{A}^1 := 0 \times_{\mathbb{A}^1} 0$.

2. Basic properties of $\mathrm{LS}_{\check{G}}(X)$

From now on, we will fix a smooth projective curve X over k. We will need two basic properties of $\mathrm{LS}_{\check{G}}(X)$: it is *quasi-compact* and *quasi-smooth*.

2.1. $\mathrm{LS}_{\check{G}}(X)$ **is quasi-compact.** There's an obvious map $\mathrm{LS}_{\check{G}}(X) \to \mathrm{Bun}_{\check{G}}(X)$ forgetting the connection (or equivalently, pre-composing with the obvious morphism $X \to X_{\mathrm{dR}}$). The morphism is quasi-compact, so if $\mathrm{Bun}_{\check{G}}(X)$ were quasi-compact, we would be done. Unfortunately, as we saw in the first talk, $\mathrm{Bun}_{\check{G}}(X)$ isn't quasi-compact. However, we do have the following.

Lemma 2.2. *Let $\check{G}$ be a linear algebraic group over k. The morphism $\mathrm{LS}_{\check{G}}(X) \to \mathrm{Bun}_{\check{G}}(X)$ lands in a quasi-compact open.*

Proof. By embedding $\check{G}$ into some GL_n, it suffices to prove the statement for $\check{G} = \mathrm{GL}_n$. We are asking which rank n vector bundles $\mathcal{E}$ admit a connection. If there were a connection then $\deg(\mathcal{E}) = 0$ so let us assume this. Recall that any rank n vector bundle $\mathcal{E}$ admits a *Harder-Narasimhan filtration*

$$0 \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_2 \subsetneq \cdots \subsetneq \mathcal{E}_r = \mathcal{E}$$

such that each subquotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable and the slopes

$$\lambda_i := \frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})}{\mathrm{rank}(\mathcal{E}_i/\mathcal{E}_{i-1})}$$

are strictly decreasing: $\lambda_1 < \lambda_2 < \cdots < \lambda_r$. Then $\mathrm{Bun}_{\mathrm{GL}_n}$ admits a *Harder-Narasimhan stratification*, whose strata are indexed by the slopes. Now note that for all but finitely many $(\lambda_i)_{i=1}^r$ we can find an index i such that $\deg(\mathcal{H}om(\mathcal{E}/\mathcal{E}_i, \mathcal{E}_i)) \gg 0$, which by Riemann-Roch applied implies $\mathrm{Ext}^1_X(\mathcal{E}/\mathcal{E}_i, \mathcal{E}_i) = 0$. So $\mathcal{E}$ splits as a direct sum of $\mathcal{E}_i \oplus \mathcal{E}/\mathcal{E}_i$, where $\deg(\mathcal{E}_i) < 0$ and $\deg(\mathcal{E}/\mathcal{E}_i) > 0$. But now $\mathcal{E}_i$ will not admit a connection since its degree is non-zero. The connection may still have some component $\mathcal{E}/\mathcal{E}_i \to \mathcal{E}_i \otimes \Omega^1_X$, but they must also vanish if $\deg(\mathcal{E}/\mathcal{E}_i) \gg \deg(\mathcal{E}_i)$. Any Harder-Narasimhan strata in $\mathrm{Bun}_{\mathrm{GL}_n}$ is quasi-compact and since the image of $\mathrm{LS}_{\mathrm{GL}_n} \to \mathrm{Bun}_{\mathrm{GL}_n}$ lands in finitely many strata, we are done. ☐

2.3. $\mathrm{LS}_{\check{G}}(X)$ **is quasi-smooth.** Quasi-smoothness is a slight weakening of smoothness which suffices for many purposes. A DG scheme Z is smooth if and only if the cotangent bundle T^*Z is a vector bundle [AG15, Lemma 2.1.2], and we weaken this condition.

Definition 2.4 ([AG15, §2.1.3, Corollary 2.1.6])**.** A DG scheme Z is *quasi-smooth* if one of the following equivalent conditions hold:

(1) the cotangent complex T^*Z is perfect of Tor-amplitude $[-1, 0]$, i.e., is Zariski-locally presented as a complex $\mathcal{O}_Z^{\oplus m} \to \mathcal{O}_Z^{\oplus n}$; or

(2) Z is Zariski-locally presented as a fiber product

$$\begin{array}{ccc} Z & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow f \\ \{0\} & \longrightarrow & \mathbb{A}^m. \end{array} \tag{2.1}$$

For reasonable stacks (see [AG15, §8.1] for a more precise definition), Z is *quasi-smooth* if the cotangent complex T^*Z lives in degree ≥ -1.

Proposition 2.5 ([AG15, Proposition 10.4.5])**.** *For any linear algebraic group $\check{G}$, the stack $\mathrm{LS}_{\check{G}}(X)$ is quasi-smooth.*

Proof. A general computation for mapping stacks shows that the tangent space at a $\check{G}$-local system $(\mathcal{P}, \nabla)$ is $R\Gamma_{\mathrm{dR}}(X, \mathfrak{g}_{\mathcal{P}})[1]$. By Verdier duality the cotangent complex at $\mathcal{P}$ is $R\Gamma_{\mathrm{dR}}(X, \mathfrak{g}^*_{\mathcal{P}})[1]$, which lives in degree ≥ -1. □

3. Singular support of coherent sheaves

Let us review a general categorical construction known as ind-completion.

Definition 3.1. For a category $\mathcal{C}$, the *ind-completion* $\mathrm{Ind}(\mathcal{C})$ has objects which are formal colimits $\mathrm{colim}_{i \in I}\, c_i$, with morphisms

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}\left(\operatorname*{colim}_{i\in I} c_i, \operatorname*{colim}_{j\in J} d_j\right) := \lim_i \operatorname*{colim}_j \mathrm{Hom}_{\mathcal{C}}(c_i, d_j).$$

Definition 3.2. For a stack Z, let $\mathrm{IndCoh}(Z) := \mathrm{Ind}(\mathrm{Coh}(Z))$.

When Z is smooth, perfect complexes and coherent sheaves match, i.e., $\mathrm{Perf}(Z) = \mathrm{Coh}(Z)$. Therefore, by passing to ind-completions we see $\mathrm{QCoh}(Z) = \mathrm{IndCoh}(Z)$. However when Z is singular, the categories are generally *not* the same.

Example 3.3. When $Z = \mathrm{Spec}\, k[\epsilon]/\epsilon^2$, the $k[\epsilon]/\epsilon^2$-module k is coherent but *not* perfect since its free resolution is infinite:

$$\cdots \xrightarrow{\epsilon} k[\epsilon]/\epsilon^2 \xrightarrow{\epsilon} k[\epsilon]/\epsilon^2 \xrightarrow{\epsilon} k[\epsilon]/\epsilon^2 \to k.$$

Example 3.4. In the setting of DG schemes, the prototypical example is $Z = 0 \times_{\mathbb{A}^1} 0 = \mathrm{Spec}\, A$ where $A = k[\epsilon]$ and $|\epsilon| = -1$. Then, under Koszul duality, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(Z) & \longrightarrow & k[\xi]\text{-}\mathrm{mod} \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(Z) & \longrightarrow & k[\xi]\text{-}\mathrm{mod}_{\mathrm{loc.nilp}}, \end{array} \tag{3.1}$$

where $|\xi| = 2$ and $k[\xi]\text{-}\mathrm{mod}_{\mathrm{loc.nilp}}$ is the sub-category of $k[\xi]\text{-}\mathrm{mod}$ consisting of $k[\xi]$-modules where ξ acts locally nilpotently. The algebra $k[\xi]$ appears since, as in usual Koszul duality, $k[\xi] = \mathrm{End}_A(k)$. The point is that there is an exact triangle of A-modules $k[1] \xrightarrow{\epsilon} A \to k$, and rotating this we obtain a A-module homomorphism $\xi \colon k \to k[2]$.

The main difference is that since $\mathrm{IndCoh}(Z) = \mathrm{Ind}(\mathrm{Coh}(Z))$ the A-module k is compact by definition. However, k isn't a compact object in $\mathrm{QCoh}(Z)$. More concretely, the colimit

$$\mathrm{colim}(k \xrightarrow{\xi} k[2] \xrightarrow{\xi} k[4] \xrightarrow{\xi} \cdots) \in \mathrm{IndCoh}(Z)$$

is zero as an A-module. This is an object in cohomological degree $-\infty$, in the sense that all of its cohomologies vanish.

Example 3.4 illustrates a general point, that generally $\mathrm{QCoh}(Z)$ and $\mathrm{IndCoh}(Z)$ are different, but their difference is only in cohomological degree $-\infty$ [Gai13, Proposition 1.2.4].

The general slogan is: *the more singular* Z *is, the farther* $\mathrm{IndCoh}(Z)$ *is from* $\mathrm{QCoh}(Z)$.

To formalize this, we first need to quantify how singular a (quasi-smooth) stack Z is. For this, recall that a stack Z is quasi-smooth when T^*Z lives in degree ≥ -1 and is smooth when T^*Z lives in degree ≥ 0. So a natural definition is to let $\mathrm{Sing}(Z)$ be the stack over Z whose fiber over $z \in Z$ is the cohomology group $H^{-1}(T^*_z Z)$.

Example 3.5. When Z is of the form (2.1), then the cotangent complex is $k^m \xrightarrow{(df)^\vee} k^n$ so $\mathrm{Sing}(Z)$ over a point $z \in Z$ is the kernel of $(df)^\vee$. This makes sense since Z is smooth exactly when df is surjective.

Arinkin-Gaitsgory [AG15] produces, for any conical subset $Y \subset \mathrm{Sing}(Z)$, a catgeory $\mathrm{IndCoh}_Y(Z)$ such that $\mathrm{IndCoh}_0(Z) = \mathrm{QCoh}(Z)$ and $\mathrm{IndCoh}_{\mathrm{Sing}(Z)}(Z) = \mathrm{IndCoh}(Z)$.

Example 3.6 ([AG15, Lemma 5.1.7])**.** Suppose $Z = 0 \times_V 0 = \mathrm{Spec}(\mathrm{Sym}(V^*[1]))$ for some k-vector space V. Then $T^*Z = V^*[1]$ so $\mathrm{Sing}(Z) = V^*$. We again have an equivalence

$$\mathrm{IndCoh}(Z) \simeq \mathrm{Sym}(V[-2])\text{-}\mathrm{mod},$$

and for a conical subset $Y \subset \mathrm{Sing}(Z) = V^*$ we let $\mathrm{IndCoh}_Y(Z)$ be the subcategory of $\mathrm{Sym}(V[-2])\text{-}\mathrm{mod}$ consisting of $\mathrm{Sym}(V[-2])$-modules M supported on Y, i.e., such that the cohomology $H^\bullet(M)$ as a graded $\mathrm{Sym}(V)$-module is supported on Y.

Generally we define singular support by interpolating from Example 3.6. Suppose Z is a DG scheme of the form (2.1). Then $\mathrm{Sing}(Z) \subset \mathbb{A}^m \times Z$ (see Example 3.5). Since $Z \times_{\mathbb{A}^n} Z \simeq \Omega\mathbb{A}^m \times Z$, we have an action map $\mathrm{act}\colon \Omega\mathbb{A}^m \times Z \to Z$. Then there is a pullback functor

$$\mathrm{act}^!\colon \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(\Omega\mathbb{A}^m) \otimes \mathrm{IndCoh}(Z) \xrightarrow{\mathrm{KD}} \mathrm{Sym}(\mathbb{A}^m[-2])\text{-}\mathrm{mod} \otimes \mathrm{IndCoh}(Z).$$

We say $\mathcal{F} \in \mathrm{IndCoh}_Y(Z)$ exactly when $\mathrm{KD}(\mathrm{act}^! \mathcal{F})$ is supported on Y.

This construction has nice functorial properties, so it can be extended to all quasi-smooth stacks.

4. Nilpotent singular support

For $\mathrm{LS}_{\check{G}}$, we computed the cotangent complex at $(\mathcal{P}, \nabla) \in \mathrm{LS}_{\check{G}}$ as $R\Gamma_{\mathrm{dR}}(X, \mathfrak{g}^*_{\mathcal{P}})[1]$. Thus, $\mathrm{Sing}(\mathrm{LS}_{\check{G}})$ classifies:

(1) a $\check{G}$-local system $(\mathcal{P}, \nabla)$ on X; and
(2) a horizontal section A of $\mathfrak{g}^*_{\mathcal{P}}$.

We denote this stack as $\mathrm{Arth}_{\check{G}}$. There is a conical subset $\mathrm{Nilp}_{glob} \subset \mathrm{Arth}_{\check{G}}$ where A is nilpotent. Now, we consider $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$. Since $\mathrm{LS}_{\check{G}}$ is quasi-compact, we know $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ is compactly generated [AG15, Corollary 4.3.2].

Why is nilpotent singular support a natural condition?

(1) Classically, in number-theoretic Langlands, there is a certain Arthur parameter $\mathrm{SL}_2 \to \check{G}$ which measures how far from tempered an automorphic representation is. By Jacobson-Morosov such a data is equivalent to the data of a nilpotent element;
(2) Eventually, we want compatibilities with *Eisenstein series*. Let $\check{P} \subset \check{G}$ be a parabolic subgroup and let $\check{M}$ be the reductive quotient of $\check{P}$. On the spectral side, there is a diagram $\mathrm{LS}_{\check{M}} \leftarrow \mathrm{LS}_{\check{P}} \to \mathrm{LS}_{\check{G}}$, and push-pull defines a functor $\mathrm{QCoh}(\mathrm{LS}_{\check{M}}) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$. The issue is that this functor *does not* preserve compact objects. If we instead consider the spectral Eisenstein series functor $\mathrm{Eis}^{spec}_{\check{P}\subset\check{G}}\colon \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$, however, this will preserve compact objects! In fact, $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ is generated by the images of $\mathrm{QCoh}(\mathrm{LS}_{\check{M}})$ under $\mathrm{Eis}^{spec}_{\check{P}\subset\check{G}}$ for all parabolics $\check{P} \subset \check{G}$.

(3) The geometric Satake equivalence states $\mathrm{Perv}(L^+G\backslash LG/L^+G) \simeq \mathrm{Rep}(\check{G})$ [MV07], but $\text{D-mod}(L^+G\backslash LG/L^+G)$ and $\mathrm{Rep}(\check{G})$ are not equivalent derived categories. Instead

$$\text{D-mod}(L^+G\backslash LG/L^+G) \simeq \mathrm{IndCoh}_{\check{\mathcal{N}}}(\Omega\check{\mathfrak{g}}/\check{G}),$$

where $\check{\mathcal{N}} \subset \check{\mathfrak{g}}$ is the nilpotent cone [BF08], [AG15, Corollary 12.5.5].

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